A construction of Hadamard cubes from association schemes on triples

Sho Suda (National Defense Academy of Japan)

Joint work with Amin Bahmanian (Illinois State University)

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Main result

A d-dimensional Hadamard matrix of order n is a d-dimensional matrix H of order n with entries $1, -1$ such that for any j and any a, b,

$$\sum_{1\leq x_1,\ldots,\hat{x_j},\ldots,x_d\leq n} H(x_1,\ldots,a,\ldots,x_d)H(x_1,\ldots,b,\ldots,x_d) = n^{d-1}\delta_{ab}.$$

A three-dimensional Hadamard matrix is said to be a Hadamard cube.
Known constructions for Hadamard cubes of order n:

- for n such that n is the order of a Hadamard matrix by Yang;
- for $n = 2 \cdot 3^k$, $k \in \mathbb{Z}_{>0}$ by Yang,
- for n such that $n-1$ is an odd prime power by Krčadinac, Pavčević, Tabak.

Main result (Bahmanian-S., 2025)

Let n be the order of a conference matrix. Then there exists a Hadamard cube of order n.

Since a conference matrix of order $n = 46$ exists, a Hadamard cube of order $n = 46$ exists, while $n - 1 = 45$ is not a prime power.

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To be more precise,

Theorem (Bahmanian-S., 2025)

(1) Let n be the order of a conference matrix. Then there exists an association scheme on triple $(X, \{R_i\}_{i=0}^5)$ with $|X| = n$.

(2) If an association scheme on triple $(X, \{R_i\}_{i=0}^5)$ with $|X| = n$ and the same parameters of that in (1) exists, then a Hadamard cube of order n exists.

Known result (Goethals-Seidel, 1970)

$\exists$ certain association schemes

$\Longrightarrow$

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Our result (Bahmanian-S., 2025)

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Contents

- Association schemes and Hadamard matrices
  - Strongly regular graph, regular symmetric Hadamard matrices with constant diagonals.
- Association schemes on triples and Hadamard cubes
  - Hadamard cubes: construction using finite fields
  - Association scheme on triples; regular two-graphs, regular skew-two graphs (the latter is new construction)
  - How to construct Hadamard cubes from association schemes on triples
  - Construction of association schemes on triples, which yield Hadamard cubes, from conference matrices.

Hadamard matrices, strongly regular graphs

- A Hadamard matrix of order n is an $n \times n$ matrix H with entries ± 1, satisfying $HH^T = nI$.
- A Hadamard matrix H of order n^2 is regular if $H\mathbf{1} = \pm n\mathbf{1}$ where $\mathbf{1}$ is the all-ones vector.

- A strongly regular graph with parameters (v, k, λ, μ) is a graph on v vertices with adjacency matrix A such that

$$A^2 = kI + \lambda A + \mu(J - I - A)$$

where I is the identity matrix and J is the all-ones matrix. Note that $AJ = JA = kI$.

Hadamard matrices and strongly regular graphs

Theorem (Goethals-Seidel, 1970)

The existence of the following are equivalent:

1. A regular symmetric Hadamard matrix of order $4t^2$ with constant diagonals.
2. A strongly regular graph with parameters $(4t^2, 2t^2 + \varepsilon t, t^2 + \varepsilon t, t^2 + \varepsilon t)$ for some $\varepsilon \in \{1, -1\}$.

- These structures are connected via the identity $H = \pm I + A - (J - I - A)$.
- A strongly regular graph is equivalent to a symmetric association scheme of class 2.
- (Real or complex) Hadamard matrices are constructed as a linear combination of adjacency matrices of association schemes by Chan-Godsil, Ikuta-Munemasa.
- Hadamard matrices with certain regularity are related to association schemes.

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Demonstration how to obtain Hadamard matrices from strongly regular graphs

- Assume that A is the adjacency matrix of an SRG;

$$A^2 = kI + \lambda A + \mu(J - I - A), \quad AJ = JA = kI.$$

- If we set $H = I + aA + b(J - I - A)$ for $a, b \in \{1, -1\}$, we can calculate as

$$\begin{aligned} HH^\top &= H^2 \\ &= (I + aA + b(J - I - A))^2 \\ &= \cdots \quad \text{(Using the above equalities)} \\ &= c_0 I + c_1 A + c_2 J \end{aligned}$$

for some $c_0, c_1, c_2, \in \mathbb{Z}$.

- Then check whether a, b exist in $\{1, -1\}$ such that $c_1 = c_2 = 0$.
$\rightarrow$ If it is yes, then H with those a, b is a Hadamard matrix.

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Hadamard cubes

A Hadamard cube of order n is an $n \times n \times n$ matrix H of order n with entries $1, -1$ such that for any a, b,

$$\sum H(x,y,a)H(x,y,b) = \sum H(x,a,y)H(x,b,y) = \sum H(a,x,y)H(b,x,y) = n^2\delta_{ab},$$

where x, y run over from 1 to n in the sum.

Krčadinac-Pavčević-Tabak, 2024:

- Let $\mathbb{F}_q$ be the finite field of order q, and χ be the quadratic character on $\mathbb{F}_q$.
- Define $H : (\mathbb{F}_q \cup \{\infty\})^3 \to \{1, -1\}$ by

$$H(x,y,z) = \begin{cases} -1 & \text{if } x = y = z, \\ 1 & \text{if } x = y \neq z \text{ or } x = z \neq y \text{ or } y = z \neq x, \\ \chi(z-y) & \text{if } x = \infty, \\ \chi(x-z) & \text{if } y = \infty, \\ \chi(y-x) & \text{if } z = \infty, \\ \chi((x-y)(y-z)(z-x)) & \text{otherwise.} \end{cases}$$

Then the three-dimensional matrix H is a Hadamard cube of order $q+1$.

Association schemes on triples

Let X be a non-empty finite set again. Let $\{R_0, \ldots, R_m\}$ be a partition of $X \times X \times X$ with $m \geq 4$. An association scheme on triples (AST) is a pair $(X, \{R_i\}_{i=0}^m)$ satisfying the following four conditions:

1. The relations R_0, R_1, R_2, R_3 are chosen so that
$$\begin{aligned} R_0 &= \{(x,x,x) \mid x \in X\}, \\ R_1 &= \{(x,y,y) \mid x,y \in X, x \neq y\}, \\ R_2 &= \{(y,x,y) \mid x,y \in X, x \neq y\}, \\ R_3 &= \{(y,y,x) \mid x,y \in X, x \neq y\}. \end{aligned}$$
2. For any $i \in \{0, \ldots, m\}$ and any permutation σ of $\{1,2,3\}$,
$$\sigma(R_i) := \{(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \mid (x_1,x_2,x_3) \in R_i\} \in \{R_0, \ldots, R_m\}$$
3. For any $i \in \{0, \ldots, m\}$, there exists $n_i \in \mathbb{Z}_{\geq 0}$ such that for any two distinct elements $x, y \in X$,
$$n_i = |\{z \in X \mid (x,y,z) \in R_i\}|.$$
4. For any $i,j,k,\ell \in \{0, \ldots, m\}$, there exists $p_{ijk}^{\ell} \in \mathbb{Z}_{\geq 0}$ such that for any $(x,y,z) \in R_\ell$,
$$p_{ijk}^{\ell} = |\{w \in X \mid (w,y,z) \in R_i, (x,w,z) \in R_j, (x,y,w) \in R_k\}|.$$
This value is said to be the intersection number.

Association schemes on triples

- For each $R_i \subset X \times X \times X$ $(i \in \{0, \ldots, m\})$, a *three-dimensional matrix* is associated as follows. Let $v = |X|$. Define a $v \times v \times v$ matrix A_i with the (x, y, z)-entry, denoted $(A_i)_{xyz}$, by

$$(A_i)_{xyz} = \begin{cases} 1 & \text{if } (x, y, z) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A_i is said to be the adjacency matrix of the hypergraph (X, R_i).

Idea to construct Hadamard cubes:
For an association scheme on triples with adjacency matrices $\{A_i\}_{0 \leq i \leq m}$, consider the following matrix

$$H = \sum_{i=0}^{m} a_i A_i$$

for some $a_i \in \{1, -1\}$.

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How to check $H = \sum_{i=0}^{m} a_i A_i$ to be a Hadamard cube

What we want to do

For distinct $x, y \in X$, to calculate the value

$$\sum_{z,w \in X} H_{xzw} H_{yzw}, \tag{1}$$

and find conditions for a_i ($i \in \{0, \ldots, m\}$) such that (1) is zero.
(Also for $\sum_{z,w \in X} H_{zxw} H_{zyw}$, $\sum_{z,w \in X} H_{zwx} H_{zwx}$.)

To calculate (1), we use the ternary product for 3-dim matrices:

- For three $v \times v \times v$ matrices A, B, and C, the ternary product for A, B, C, denoted ABC is the $v \times v \times v$ matrix D whose (x, y, z)-entry given by

$$(D)_{xyz} = \sum_{w \in X} (A)_{wyz} (B)_{xwz} (C)_{xyw}.$$

For an AST $(X, \{R_i\}_{i=0}^{m})$, the intersection numbers

$$p_{ijk}^{\ell} = |\{w \in X \mid (w, y, z) \in R_i, (x, w, z) \in R_j, (x, y, w) \in R_k\}|$$

appear in the ternary product: $A_i A_j A_k = \sum_{\ell=0}^{m} p_{ijk}^{\ell} A_\ell.$

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- By combining (2) and (3), obtain $\sum_{w \in X} H_{ywz} H_{xwz} = \sum_{i,j,k,\ell=0}^{m} b_i a_j p_{ijk}^{\ell} (A_\ell)_{xyz}$.

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- $$\sum_{w \in X} H_{ywz} H_{xwz} = \sum_{i,j,k,\ell=0}^{m} b_i a_j p_{ijk}^{\ell} (A_\ell)_{xyz}. \tag{4}$$

- Take a sum in (4) over $z \in X$ to obtain

$$\sum_{z,w \in X} H_{xzw} H_{yzw} = \sum_{i,j,k,\ell=0}^{m} b_i a_j p_{ijk}^{\ell} \sum_{z \in X} (A_\ell)_{xyz}$$

- Then

$$\sum_{z \in X} (A_\ell)_{xyz} = |\{z \in X \mid (x,y,z) \in R_\ell\}| = n_\ell. \quad \text{(by the definition 3 in AST)}$$

Therefore, we obtain:

$$\sum_{z,w \in X} H_{xzw} H_{yzw} = \sum_{i,j,k,\ell=0}^{m} b_i a_j p_{ijk}^{\ell} n_\ell.$$

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- Take a sum in (4) over $z \in X$ to obtain

$$\sum_{z,w \in X} H_{xzw} H_{yzw} = \sum_{i,j,k,\ell=0}^{m} b_i a_j p_{ijk}^{\ell} \sum_{z \in X} (A_\ell)_{xyz}$$

- Then

$$\sum_{z \in X} (A_\ell)_{xyz} = |\{z \in X \mid (x,y,z) \in R_\ell\}| = n_\ell.$$ (by the definition 3 in AST)

Therefore, we obtain:

$$\sum_{z,w \in X} H_{xzw} H_{yzw} = \sum_{i,j,k,\ell=0}^{m} b_i a_j p_{ijk}^{\ell} n_\ell.$$

How to check $H = \sum_{i=0}^{m} a_i A_i$ to be a Hadamard cube

- For distinct $x, y \in X$, we want to calculate the value
$$\sum_{z,w \in X} H_{xzw} H_{yzw}. \tag{1}$$
- $$\sum_{w \in X} H_{ywz} H_{xwz} = \sum_{i,j,k,\ell=0}^{m} b_i a_j p_{ijk}^{\ell} (A_\ell)_{xyz}. \tag{4}$$
- Take a sum in (4) over $z \in X$ to obtain
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$$\sum_{z,w\in X} H_{xzw} H_{yzw} = \sum_{i,j,k,\ell=0}^{m} b_i a_j p_{ijk}^{\ell} n_{\ell}. \quad (5)$$

- Note that p_{ijk}^{ℓ} and n_{ℓ} in RHS in (5) are the parameters of AST.
- $(b_0, \ldots, b_m)$ is a permutation of $(a_0, \ldots, a_m)$, depending on the AST.
- The remaining thing to do is to check which $a_j, b_i \in \{1, -1\}$ $(i, j \in \{0, \ldots, m\})$ satisfy that RHS in (5) is zero.
- Similar to the cases of $\sum_{z,w\in X} H_{zxw} H_{zyw}$, $\sum_{z,w\in X} H_{zwx} H_{zwy}$.

$\rightarrow$ Then, which AST do we consider?

Examples of AST

Examples of ASTs:

- 2-$(v, k, 1)$ designs;
- 2-transitive groups;
- regular two-graphs (including an example obtained from symmetric conference matrices);
- $\dots$
- regular skew two-graphs (that are obtained from skew-symmetric conference matrices) (new example)

The rest of this talk describes how ASTs are obtained from conference matrices.

AST from conference matrices

- A conference matrix of order n is an $n \times n$ $(0, 1, -1)$-matrix C with zero diagonal entries such that $CC^\top = (n-1)I$. (n must be even.)
- For some diagonal matrices D, D' with diagonal entries $1, -1$,
  - DCD' is symmetric if $n = 4k+2$, $k \in \mathbb{Z}_{>0}$;
  - DCD' is skew-symmetric if $n = 4k$, $k \in \mathbb{Z}_{>0}$.
- Conference matrices of order n exist for, among others, the following values:
  - $n = q+1$ where q is an odd prime number,
  - $n = q^2(q+2)+1$ where $q \equiv 3 \pmod 4$ is a prime power and $q+2$ is a prime power,
  - $n = 5 \cdot 9^{2t+1} + 1$ where t is a non-negative integer,
  - $\cdots$.

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Let C be a symmetric conference matrix of order n or skew-symmetric conference matrix of order n. Define $X = \{1, \dots, n\}$ and

$$R_4 = \{(x,y,z) \in X \times X \times X \mid C_{xy}C_{yz}C_{zx} = 1\},$$

$$R_5 = \{(x,y,z) \in X \times X \times X \mid C_{xy}C_{yz}C_{zx} = -1\}.$$

Theorem (Mesner-Bhattacharya, 1990, Bahmanian-S., 2025)

The pair $(X, \{R_i\}_{i=0}^5)$ is an association scheme on triples.

The result for symmetric conference matrix case is due to Mesner-Bhattacharya and that for skew-symmetric conference matrix case is our new result.

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Theorem 1 (Mesner-Bhattacharya, 1990, Bahmanian-S., 2025)

The pair $(X, \{R_i\}_{i=0}^5)$ is an association scheme on triples.

Theorem 2 (Bahmanian-S., 2025)

Let $A_0, \ldots, A_5$ be the adjacency matrices of the AST in Theorem 1. Then $H = \sum_{i=0}^5 a_i A_i$ is a Hadamard cube of order n if and only if $(a_i)_{i=0}^5 \in \{1, -1\}^6$ satisfies $a_0a_1a_2a_3 = a_4a_5 = -1$.

Proof: Check $\sum_{z,w \in X} H_{xzw}H_{yzw} = \sum_{i,j,k,\ell=0}^m b_i a_j p_{ijk}^\ell n_\ell = 0$. Similar to the other cases.

Main result

Theorem (Bahmanian-S., 2025)

(1) Let n be the order of a conference matrix C. Then there exists an association scheme on triple $(X, \{R_i\}_{i=0}^5)$ with $|X| = n$.

(2) Let $A_0, \ldots, A_5$ be the adjacency matrices of the AST in (1). Then $H = \sum_{i=0}^5 a_i A_i$ is a Hadamard cube of order n if and only if $(a_i)_{i=0}^5 \in \{1, -1\}^6$ satisfies $a_0 a_1 a_2 a_3 = a_4 a_5 = -1$.

- If $C = \begin{bmatrix} 0 & \mathbf{1}^\top \\ \pm\mathbf{1} & \chi(x-y) \end{bmatrix}$ where χ is the quadratic character of $\mathbb{F}_q$ and $x, y \in \mathbb{F}_q$, and $(a_i)_{i=0}^5 = (-1, 1, 1, 1, 1, -1)$, the resulting Hadamard cube is the same as the one by Krčadinac, Pavčević, Tabak.
- A conference matrix of order $n = 46$ and thus a Hadamard cube of order $n = 46$ exists, while $n - 1 = 45$ is not a prime power.

Remark

- ASTs are constructed from any regular two-graphs, that is, Seidel matrices with only two-distinct eigenvalues, say ρ_1, ρ_2. For those ASTs with eigenvalues $\rho_1 + \rho_2 = \pm 2$, Hadamard cubes are constructed in the same way.

- "Weighing cubes" or "complex Hadamard cubes" may be constructed in this way.

Conclusion

Research Problems:

- No conference matrix exists for order $n = 22, 34$, or 58. How about existence of Hadamard cubes of these orders? Is there any non-existence result for Hadamard cubes?
- It would be interesting to check whether any other AST yields a Hadamard cube or not.
- Define "association schemes on d-tuples" and find an example d-dimensional Hadamard matrices which is a linear combination of the adjacency matrices.

Thank you for your attention!

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Thank you for your attention!