Switching for 2-designs and applications

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The outline of the talk:

1. Introduction and motivation
2. Switching for 2-designs
3. Applications

Switching in a (simple) graph means reversing the adjacencies of some pairs of vertices, so that an adjacent pair becomes nonadjacent and a nonadjacent pair becomes adjacent.

- Switching methods have been successfully used for constructing and studying strongly regular graphs - Seidel switching, Godsil-McKay (GM) switching, Wang-Qiu-Hu (WQH) switching.

- J. J. Seidel, Graphs and two-graphs, in: Proceedings of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1974), Congressus Numerantium X, Utilitas Math., Winnipeg, Man., 1974, 125–143.
- C. D. Godsil, B. D. McKay, Constructing cospectral graphs, Aequationes Math. 25 (1982), 257–268.
- W. Wang, L. Qiu and Y. Hu, Cospectral graphs, GM-switching and regular rational orthogonal matrices of level p, Lin. Alg. Appl. 563 (2019), 154177.

Switching methods for **strongly regular graphs** have been used for example:

- A. Abiad, S. Butler, W. H. Haemers, Graph switching, 2-ranks, and graphical Hadamard matrices, Discrete Math. 342 (2019), 2850–2855.
- M. Behbahani, C. Lam, P. R. J. Östergård, On triple systems and strongly regular graphs, J. Combin. Theory Ser. A 119 (2012), 1414–1426.

Denniston used a method called switching ovals for a construction of **symmetric** $(25, 9, 3)$ **designs**.

- R. H. F. Denniston, Enumeration of symmetric designs (25,9,3), in: Algebraic and geometric combinatorics, North-Holland Math. Stud. 65, North-Holland, Amsterdam, 1982, 111–127.

Orrick defined switching operations for **Hadamard matrices**.

- W. P. Orrick, Switching operations for Hadamard matrices, SIAM J. Discrete Math. 22 (2008), 31–50.

The switching using Pasch configurations, so called Pasch switch, was used for a construction of new **Steiner triple systems** from known ones.

- P. B. Gibbons, Computing techniques for the construction and analysis of block designs, Techn. Rept. # 92, Dept. Computer Sci., Univ. Toronto 92 (1976).
- M. J. Grannell, T. S. Griggs, Pasch configuration, in: M. Hazewinkel (Ed.), Encyclopaedia of Mathematic, Supplement III, Kluwer Academic Publishers, 2001, 299–300.

Östergård introduced a switching for **codes** and **Steiner systems**.

- P. R. J. Östergård, Switching codes and designs, Discrete Math. 312 (2012), 621–632.

Norton, Parker and Wanless used switching for a construction of **Latin squares**.

- H. W. Norton, The 7×7 squares, Ann. Eugenics 9 (1939), 269–307.
- E. T. Parker, Computer investigation of orthogonal Latin squares of order ten, in: Proc. Sympos. Appl. Math. vol. XV, Amer. Math. Soc., Providence, 1963, 73–81.
- I. M. Wanless, Cycle switches in Latin squares, Graphs Combin. 20 (2004), 545–570.

Jungnickel and Tonchev used maximal arcs for a transformation of **quasi-symmetric designs** that leads to a construction of new quasi-symmetric designs i.e. switching.

- D. Jungnickel, V. D. Tonchev, Exponential Number of Quasi-Symmetric SDP Designs and Codes Meeting the Grey-Rankin Bound, Des. Codes Cryptogr. 1 (1991), 247–253.

We introduced a switching that can be applied to **2-designs** having a set of blocks that satisfy certain conditions and show that in some cases the switching can be **directly apply** to orbit matrices.

- D. Crnković, A. Švob, Switching for 2-designs, Des. Codes Cryptogr. 90 (2022), 1585–1593.

A t-(v, k, λ) **design** is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

Every element of $\mathcal{P}$ is incident with exactly r elements of $\mathcal{B}$.

The number of blocks is denoted by b.

If $b = v$ (or equivalently $k = r$) then the design is called **symmetric**.

- A 2-(v, k, λ) design is called a block design.
- If $\mathcal{D}$ is a t-design, then it is also a s-design, for $1 \leq s \leq t-1$.
- An **incidence matrix** of a design $\mathcal{D}$ is a matrix $A = [a_{ij}]$ where $a_{ij} = 1$ if jth point is incident with the ith block and $a_{ij} = 0$ otherwise.

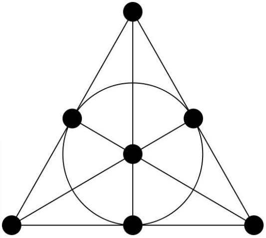

Figure: 2-$(7, 3, 1)$ design

Switching set

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a 2-design and let $\mathcal{B}_1 \subset \mathcal{B}$ be a set of blocks such that there are sets of points $\mathcal{P}_1, \mathcal{P}_2 \subset \mathcal{P}$ with the following properties:

1. $(P, B) \notin \mathcal{I}$, for every $(P, B) \in \mathcal{P}_1 \times \mathcal{B}_1$,
2. $(P, B) \in \mathcal{I}$, for every $(P, B) \in \mathcal{P}_2 \times \mathcal{B}_1$,
3. $|\{B \in \mathcal{B}_1 : (P, B) \in \mathcal{I}\}| = |\{B \in \mathcal{B}_1 : (P, B) \notin \mathcal{I}\}|$, for every $P \in \mathcal{P} \backslash (\mathcal{P}_1 \cup \mathcal{P}_2)$.

Then $\mathcal{B}_1$ is called a **switching set** of $\mathcal{D}$.

$b \times v$ matrix:

$$\left[\begin{array}{c|c|c} & & \\ \hline 0 & 1 & 0,1 \end{array}\right]$$

If $\mathcal{B}_1$ is a switching set of a 2-(v,k,λ) design $\mathcal{D}=(\mathcal{P},\mathcal{B},\mathcal{I})$, we define an incidence structure $\mathcal{D}_1=(\mathcal{P},\mathcal{B},\mathcal{I}_1)$ obtained from $\mathcal{D}$ by switching with respect to $\mathcal{B}_1$ in the following way:

1. $(P,B)\in\mathcal{I}_1 \Leftrightarrow (P,B)\in\mathcal{I}$, for $B\in\mathcal{B}\backslash\mathcal{B}_1$, $P\in\mathcal{P}$,
2. $(P,B)\in\mathcal{I}_1 \Leftrightarrow (P,B)\in\mathcal{I}$, for $B\in\mathcal{B}_1$, $P\in\mathcal{P}_1\cup\mathcal{P}_2$,
3. $(P,B)\in\mathcal{I}_1 \Leftrightarrow (P,B)\notin\mathcal{I}$, for $B\in\mathcal{B}_1$, $P\in\mathcal{P}\backslash(\mathcal{P}_1\cup\mathcal{P}_2)$.

$b\times v$ matrix:

$$\left[\begin{array}{c|c|c} & & \\ \hline 0 & 1 & 1,0 \end{array}\right]$$

Theorem 1 [D. Crnković, AŠ]

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a 2-(v, k, λ) design. If $\mathcal{B}_1$ is a switching set of $\mathcal{D}$ then the incidence structure $\mathcal{D}_1 = (\mathcal{P}, \mathcal{B}, \mathcal{I}_1)$ obtained from $\mathcal{D}$ by switching with respect to $\mathcal{B}_1$ is also a 2-(v, k, λ) design.

If a design $\mathcal{D}_1$ is obtained from $\mathcal{D}$ by switching with respect to $\mathcal{B}_1$, then $\mathcal{D}$ can be obtained form $\mathcal{D}_1$ also by switching with respect to $\mathcal{B}_1$. If 2-designs $\mathcal{D}$ and $\mathcal{D}_1$ can be obtained from each other by switching, then $\mathcal{D}$ and $\mathcal{D}_1$ are said to be **switching-equivalent**.

If $\mathcal{B}_1$ is a switching set of a symmetric 2-design $\mathcal{D}$, then the incident structure with the point set $\mathcal{B}_1$ and the block set $\mathcal{P} \backslash (\mathcal{P}_1 \cup \mathcal{P}_2)$ is a 2-design which is a subdesign of the dual design of $\mathcal{D}$.

Remark: If $\mathcal{D}$ is a block design, the incidence structure $\mathcal{D}'$ having as points the blocks of $\mathcal{D}$, and having as blocks the points of $\mathcal{D}$, where a point and a block are incident in $\mathcal{D}'$ if and only if the corresponding block and a point of $\mathcal{D}$ are incident, is a block design called the dual of $\mathcal{D}$. The dual design of a 2-design $\mathcal{D}$ is a 2-design if and only if $\mathcal{D}$ is symmetric.

The switching introduced in this talk defines a trade.

A **trade** for a 2-(v, k, λ) design consists of two disjoint sets of blocks with the property that if the design contains the blocks of one of the sets, then these blocks can be replaced by the blocks of the other set.

For example, if a design has a subdesign, then the subdesign can be replaced by a disjoint subdesign with the same parameters.

A **Hadamard matrix** of order m is an $(m \times m)$-matrix $H = (h_{i,j})$, $h_{i,j} \in \{-1, 1\}$, satisfying $HH^T = H^T H = mI_m$, where I_m is the unit matrix of order m.

A Hadamard matrix is **regular** if the row and column sums are constant. If H is a regular Hadamard matrix, then the order of H is $4n^2$.

Symmetric (64,28,12) designs are related to regular Hadamard matrices of order 64.

By applying Theorem 1 we obtained 3 new symmetric $(64, 28, 12)$ designs (without use of computer) from designs constructed in the paper:

- D. Crnković, M.-O. Pavčević, Some new symmetric designs with parameters (64,28,12), Discrete Math. 237, No. 1-3 (2001), 109–118.

The switching sets were determined by orbits of the group Z_7 i.e. producing the orbit matrix M_3'.

- Any of the last five orbits for Z_7 of the designs $\mathcal{D}_{27}$, $\mathcal{D}_{28}$ and $\mathcal{D}_{29}$ from the paper by D. Crnković and M.-O. Pavčević, together with the fixed block, form a switching set of size 8.

M_3'	1	7	7	7	7	7	7	7	7	7
1	0	7	7	7	7	0	0	0	0	0
7	1	4	4	4	0	3	3	3	3	3
7	1	4	4	0	4	3	3	3	3	3
7	1	4	0	4	4	3	3	3	3	3
7	1	0	4	4	4	3	3	3	3	3
7	0	3	3	3	3	4	0	4	4	4
7	0	3	3	3	3	0	4	4	4	4
7	0	3	3	3	3	4	4	4	4	0
7	0	3	3	3	3	4	4	0	4	4
7	0	3	3	3	3	4	4	4	0	4

- By switching, from each of the designs $\mathcal{D}_{27}$, $\mathcal{D}_{28}$ and $\mathcal{D}_{29}$ we obtain, up to isomorphism, one new design denoted by $\mathcal{D}'_{27}$, $\mathcal{D}'_{28}$ and $\mathcal{D}'_{29}$, respectively, which are pairwise non-isomorphic.

Remarks:

- While the designs $\mathcal{D}_{27}$, $\mathcal{D}_{28}$ and $\mathcal{D}_{29}$ are self-dual, the newly obtained designs $\mathcal{D}'_{27}$, $\mathcal{D}'_{28}$ and $\mathcal{D}'_{29}$ are not self-dual, and together with their duals give us six designs that are not isomorphic to the designs obtained before.
- The full automorphism group of $\mathcal{D}'_{27}$ is isomorphic to $Z_7 \times Z_2$, and the full automorphism groups of $\mathcal{D}'_{28}$ and $\mathcal{D}'_{29}$ are isomorphic to Z_7.
- While the designs $\mathcal{D}_{27}$, $\mathcal{D}_{28}$ and $\mathcal{D}_{29}$ have 2-rank equal to 26, the 2-rank of any of the design $\mathcal{D}'_{28}$ and $\mathcal{D}'_{29}$ is 27.

A **Bush-type Hadamard matrix** of order $4n^2$ is a Hadamard matrix with the additional property of being a block matrix $H = [H_{i,j}]$ with blocks of size $2n \times 2n$, such that $H_{i,i} = J_{2n}$ and $H_{i,j}J_{2n} = J_{2n}H_{i,j} = 0,\ i \neq j$, $1 \leq i \leq 2n,\ 1 \leq j \leq 2n$, where J_{2n} is the all-ones $(2n \times 2n)$-matrix.

A Hadamard matrix is regular if the row and column sums are constant. If H is a regular Hadamard matrix, then the order of H is $4n^2$. Obviously, Bush-type Hadamard matrices are regular.

- H. Kharaghani showed that a Bush-type Hadamard matrix of order $4n^2$ with $2n-1$ or $2n+1$ a prime power, can be used to construct infinite classes of symmetric designs.
- Janko and Kharaghani constructed strongly regular graphs with parameters $(936, 375, 150, 150)$ and $(1800, 1029, 588, 588)$ from a block negacyclic Bush-type Hadamard matrix of order 36.
- H. Kharaghani showed that Bush-type Hadamard matrices of order $16n^2$ exist for all values of n for which a Hadamard matrix of order $4n$ exists.
- M. Muzychuk and Q. Xiang gave a construction of Bush-type Hadamard matrices of order $4n^4$ for any odd n.

It is very difficult to decide whether Bush-type Hadamard matrices of order $4n^2$ exist if n is an odd prime.
Bush-type Hadamard matrices of order $4n^2$, where n is an odd prime, have been constructed for $n = 3, 5$.

- Z. Janko, The existence of a Bush-type Hadamard matrix of order 36 and two new infinite classes of symmetric designs, J. Combin. Theory Ser. A 95 (2001), 360–364.
- Z. Janko, H. Kharaghani, A block negacyclic Bush-type Hadamard matrix and two strongly regular graphs, J. Combin. Theory Ser. A 98 (2002), 118–126.
- Z. Janko, H. Kharaghani, V. D. Tonchev, Bush-type Hadamard matrices and symmetric designs, J. Combin. Des. 9 (2001), 72–78.
- D. Crnković, D. Held, Some new Bush-type Hadamard matrices of order 100 and infinite classes of symmetric designs, J. Combin. Math. Combin. Comput. 47 (2003), 155–164.

The switching introduced in this talk can be applied to **any symmetric design obtained from a Bush-type Hadamard matrix**.

- **The six diagonal blocks** of the block negacyclic Bush-type Hadamard matrix of **order 36** constructed by Janko and Kharaghani determine six switching sets of the corresponding Menon design, and switching leads us to 64 pairwise non-isomorphic symmetric $(36, 15, 6)$ designs (including the starting one).
- These 64 designs correspond to 14 equivalence classes of Bush-type Hadamard matrices of order 36.

- **The ten diagonal blocks** of the Bush-type Hadamard matrix constructed by Janko, Kharaghani and Tonchev determine ten switching sets of the corresponding design, and switching leads us to 1024 symmetric $(100, 45, 20)$ designs (including the starting one).
- In total, 208 of these 1024 designs are pairwise non-isomorphic, leading to 120 equivalence classes of Bush-type Hadamard matrices of order 100.

Conclusion:

- we obtain six new symmetric (64,28,12) designs,
- we construct 86 pairwise non-isomorphic symmetric $(36, 15, 6)$ designs leading to 28 new pairwise nonequivalent Bush-type Hadamard matrices of order 36,
- we construct 207 pairwise non-isomorphic symmetric $(100, 45, 20)$ designs leading to 119 pairwise nonequivalent Bush-type Hadamard matrices of order 100.

Remarks:

- Examples show that the switching does not preserve the p-rank of a 2-design (in case when p divides the order of the design), and also does not preserve the self-duality of a symmetric design.
- The switching does not preserve the action of an automorphism group of a 2-design.

Muchas gracias por su atención!