Circulant complex Cretan matrices

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Cretan matrices

A **Cretan matrix** is a square matrix S such that

1. its elements satisfy

$$-1 \leq s_{ij} \leq 1 \qquad \forall i, j\,;$$

2. at least one element in each row and column is equal to 1;
3. and

$$SS^T = \omega I \qquad \text{for some } \omega.$$

If the elements of S have τ different values (levels), S is called a τ-level Cretan matrix.

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Remark. Cretan matrices with a small number of levels are of interest.

Remark. The notion *Cretan matrix* was introduced after a conference in Crete in 2014 (N. A. Balonin, M. B. Sergeev, J. Seberry).

2-level Cretan matrices

Theorem (Seberry and Balonin 2015). The existence of a 2-level Cretan matrix of order $4n-1$ is equivalent to the existence of a Hadamard matrix of order $4t$.

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Proof (sketch).

Hadamard matrix H of order 4t

$$\updownarrow$$

$$\underbrace{\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}}_{P} H \underbrace{\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}}_{P} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & B & \\ 1 & & & \end{pmatrix}$$

B - core of H

$$\updownarrow$$

Replacing 1's and -1's in B by values $y = \frac{-t+\sqrt{t}}{t-1}$ and 1, respectively, gives a Cretan matrix.

Cretan matrices

- **Hadamard matrices**

$$h_{ij} \in \{1, -1\} \quad \text{for all } i, j$$
$$HH^T = nI$$

- **conference matrices**

$$c_{ij} = \begin{cases} 0 & \text{for } i = j \\ \pm 1 & \text{for } i \neq j; \end{cases}$$
$$CC^T = (n-1)I$$

- **weighing matrices**

$$w_{ij} \in \{0, 1, -1\} \quad \text{for all } i, j$$
$$WW^T = wI \qquad (w \text{ – weight of } W)$$

Complex Cretan matrices

A **complex Cretan matrix** is a square matrix S such that

1. $|s_{ij}| \leq 1$ for all i, j;
2. at least one element in each row and column has modulus 1;
3. $SS^T = \omega I$ for some ω.

- complex Hadamard matrices: $|h_{ij}| = 1$ for all i, j;

$$HH^* = nI$$

- complex conference matrices: $|c_{ij}| = \begin{cases} 0 & \text{for } i = j; \\ 1 & \text{for } i \neq j; \end{cases}$

$$CC^* = (n-1)I$$

Circulant matrix

A **circulant matrix** is a square matrix of the form

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-2} \\ \vdots & c_{n-1} & c_0 & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}$$

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Notation:

$$C = \mathrm{circ}_n(c_0, c_1, \ldots, c_{n-2}, c_{n-1})$$

Circulant matrices with real entries

Hadamard circulant matrices

Hadamard circulant conjecture (Ryser 1963):

Hadamard circulant matrices exist only of order $n = 4$ and (trivially) $n = 1$.

Examples:

$$\pm\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}, \qquad \pm\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

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Hadamard circulant conjecture is known to be true for **symmetric matrices** (Johnsen 1964, Brualdi and Newman 1965, McKay and Wang 1987, Craigen and Kharagani 1993).

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Remark. A similar statement is proved for complex **Hermitian matrices** with entries in $\{1, -1, \mathrm{i}, -\mathrm{i}\}$ (Craigen and Kharagani 1993).

Circulant conference matrices

Theorem (Stanton and Mullin 1976).

A circulant conference matrix, i.e.,

$$C = \mathrm{circ}_n(0, \pm 1, \pm 1, \ldots, \pm 1) \qquad (n > 1)$$

such that

$$CC^T = (n-1)I$$

exists only for $n = 2$.

Solutions: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

General diagonal

$$C = \begin{pmatrix} d & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & d & c_1 & & c_{n-2} \\ \vdots & c_{n-1} & d & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & d \end{pmatrix} = \mathrm{circ}_n(d, c_1, \ldots, c_{n-1})$$

with $n > 1$ such that

$$\begin{aligned} c_j &\in \{c, -c\} && \text{for all } j = 1, \ldots, n-1; \\ CC^T &= \omega I && (\omega = d^2 + (n-1)c^2). \end{aligned}$$

...a 3-level (or 2-level) Cretan matrix

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Remark. $d = 0$, $d = 1$: circulant conference/Hadamard matrix

General diagonal

Theorem (T. and Goyeneche 2019). Consider

$$C = \mathrm{circ}_n(d, \pm 1, \pm 1, \dots, \pm 1)$$

with $n > 1$ such that

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- $2d$ is odd $\quad \Rightarrow \quad C$ exists iff $\underline{n = 2d + 2}$
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  (orthogonality $\Rightarrow$ C is symmetric, $n = 2d + 2 = 0$)

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- d is even $\quad \Rightarrow \quad C$ exists iff $\underline{n = 2d + 2}$
  (orthogonality $\quad \Rightarrow \quad C$ is symmetric, $n = 2d + 2 = 0$)

Remark. Choice $d = 0$ gives the result by Stanton and Mullin.

General diagonal: symmetric C

Theorem (T. and Goyeneche 2019). A symmetric circulant matrix

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4. $k < 2^7$: $\quad k + 1 \leq 2^r$ is satisfied in only 2 cases:
   $k = 7, n = 120; \quad k = 13, n = 924$

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5. $k = 7, n = 120$: no solution
   $k = 13, n = 924$: no solution

General diagonal: conjecture

Unresolved case: d is odd and C is not symmetric

Conjecture. Let

$$C = \mathrm{circ}_n(d, \pm 1, \pm 1, \ldots, \pm 1)$$

with $n > 1$ such that

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For any $d \geq 0$, C exists only for $n = 2d + 2$.

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Remark. The conjecture is in agreement with results for even d, for symmetric C, and with Hadamard circulant conjecture ($d = 1 \Rightarrow n = 4$).

General diagonal: all solutions for $n = 2d + 2$

We don't know whether there is a solution with $n \neq 2d + 2$.
– If there is, d must be odd and C cannot be symmetric.

However, we have a complete description of matrices satisfying $n = 2d + 2$.

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However, we have a complete description of matrices satisfying $n = 2d + 2$.

Theorem (T. and Goyeneche 2019). Every real matrix

$$C = \mathrm{circ}_n(d, \pm 1, \pm 1, \ldots, \pm 1)$$

such that

$$CC^T = (d^2 + n - 1)I$$

has its first row of one of the forms below:

$$\begin{aligned}
&(d, -1, -1, \ldots, -1)\\
&(d, 1, -1, 1, -1, \ldots, 1, -1, 1)\\
&(d, 1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1, 1, -1)\\
&(d, -1, 1, 1, -1, -1, 1, 1, \ldots, -1, -1, 1, 1)
\end{aligned}$$

Matrices with complex entries

2- and 3-level Hermitian circulant matrices

Remark. 1-level: (1), (-1)

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Observation. All Hermitian circulant 2-level Cretan matrices are real.

Proof. $c_j \notin \mathbb{R} \quad \Rightarrow \quad$ at least 3 levels: c_0, c_j, $\overline{c_j}$

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Observation. Let $n = 4k$. The matrix

$$C = \mathrm{circ}_n(c_0, c_1, \ldots, c_{n-1})$$

with the entries

$$c_j = \begin{cases} \frac{2}{n} & \text{for } j \neq \frac{n}{4} \text{ and } j \neq \frac{3n}{4} \\ \frac{4-n}{2n} + \frac{\mathrm{i}}{2} & \text{for } j = \frac{n}{4} \\ \frac{4-n}{2n} - \frac{\mathrm{i}}{2} & \text{for } j = \frac{3n}{4} \end{cases}$$

is a (multiple of a) 3-level Hermitian Cretan matrix.

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Remark. Other examples of 3-level Hermitian circulant Cretan matrices can be constructed from combinatorial designs. (Full classification is in progress.)

Complex matrices with fourth roots of unity

Theorem (Craigen and Kharaghani 1993). A Hermitian circulant complex Hadamard matrix with entries in $\{1, -1, \mathrm{i}, -\mathrm{i}\}$ exists only of order $n = 4$ (and $n = 1$).

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Theorem (Uzcátegui Contreras et al. 2021). Consider a Hermitian circulant matrix

$$C = \mathrm{circ}_n(d, c_1, c_2, \ldots, c_{n-1}) \quad (n > 1)$$

with $c_j \in \{1, -1, \mathrm{i}, -\mathrm{i}\}$ for all $j = 1, \ldots, n-1$ such that

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We have:

- d is not integer $\Rightarrow$ C exists only of order $n = 2d + 2$;

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We have:

- d is not integer $\Rightarrow$ C exists only of order $n = 2d + 2$;
- d is even $\Rightarrow$ C exists only of order $n = 2d + 2$.

Moreover, C is real and takes one of the two forms below:

$$\mathrm{circ}_n(d, -1, -1, \dots, -1)$$
$$\mathrm{circ}_n(d, 1, -1, 1, -1, \dots, 1, -1, 1)$$

Complex matrices with fourth roots of unity

Theorem (Uzcátegui Contreras et al. 2021). Let d be odd. If the conjecture

"A circulant matrix $C = \mathrm{circ}_n(d, \pm 1, \pm 1, \ldots, \pm 1, \pm 1)$ with $n > 1$ such that $CC^T = (d^2 + n - 1)I$ exists only if $n = 2d + 2$"

is true, then a Hermitian circulant matrix $C = \mathrm{circ}_n(d, c_1, c_2, \ldots, c_{n-1})$ with $n > 1$ such that

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exists only of order $n = 2d + 2$. Moreover, C takes one of the forms below:

$$\begin{gathered}
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\mathrm{circ}_n(d, \mathrm{i}, 1, -\mathrm{i}, -1, \mathrm{i}, 1, -\mathrm{i}, -1, \ldots, \mathrm{i}, 1, -\mathrm{i}) \\
\mathrm{circ}_n(d, -\mathrm{i}, 1, \mathrm{i}, -1, -\mathrm{i}, 1, \mathrm{i}, -1, \ldots, -\mathrm{i}, 1, \mathrm{i})
\end{gathered}$$

References

- R.A. Brualdi: A note on multipliers of difference sets. *J. Res. Natl. Bur. Stand.* **69B** (1965) 87–89.
- R. Craigen, H. Kharaghani: On the nonexistence of Hermitian circulant complex Hadamard matrices. *Australas. J. Combin.* **7** (1993) 225–227.
- E.C.Johnsen: The inverse multiplier for abelian group difference sets. *Canad. J. Math.* **16** (1964) 787–796.
- J.H. McKay, S.S.-S. Wang: On a theorem of Brualdi and Newman. *Linear Algebra Appl.* **92** (1987) 39–43.
- R.G. Stanton, R.C. Mullin: On the nonexistence of a class of circulant balanced weighing matrices. *SIAM J. Appl. Math.* **30** (1976) 98–102.
- O. Turek, D. Goyeneche: A generalization of circulant Hadamard and conference matrices. *Linear Algebra Appl.* **569** (2019) 241–265.
- D. Uzcátegui Contreras, D. Goyeneche, O. Turek, Z. Václavíková: Circulant matrices with orthogonal rows and off-diagonal entries of absolute value 1, *Communications in Mathematics* **29** (2021), 15–34.
- A. Klézlová: Unitary circulant matrices (Bachelor thesis).University of Ostrava (2024).

Thank you for your attention!