Aalto University
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# Non-affine Families of 8 x 8 Complex Hadamard Matrices

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# Complex Hadamard Matrices

- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
- $H \in M_n(\mathbb{T})$ is called a *complex Hadamard matrix* of order $n$ if it satisfies the equation $HH^\dagger = nI$.
- Complex Hadamard matrices of order $n$ is denoted by $\mathbb{H}(n)$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

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- Complex Hadamard matrices of order $n$ is denoted by $\mathbb{H}(n)$.
- $H_1, H_2 \in \mathbb{H}(n)$ are *equivalent* if $H_1 = D_1 P_1 H_2 P_2 D_2$ and *permutation equivalent* if $H_1 = P_1 H_2 P_2$.
- $H \in \mathbb{H}(n)$ is called *dephased* if its first row and first column consist only of ones.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

# Families of Complex Hadamard Matrices

- Let $f : D \to \mathbb{H}_d(n)$ be a continuous function, where $D \subseteq \mathbb{T}^k$.
- We call the set $H_n^{(k)} := f(D)$ a *k-parameter family of complex Hadamard matrices*.
- $H_n^{(k)}$ is called an *affine family* if $f$ can be given in the form $f(x_1, \dots, x_k) = [c \cdot x_1^{\alpha_1} \cdots x_k^{\alpha_k}]_{i,j}$, where $\alpha_i \in \mathbb{Z}$ and $c \in \mathbb{T}$.
- $H \in H_n^{(k)}$ is used as a notation for $\exists H' \in H_n^{(k)}$ s.t. $H \cong H'$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & ia & -1 & -ia \\ 1 & -1 & 1 & -1 \\ 1 & -ia & -1 & ia \end{bmatrix}$$

# Known Complex Hadamard Matrices of Order 8

- Affine families:

$$F_8^{(5)},\ D_{8A}^{(5)},\ D_{8B}^{(5)},\ S_{8A}^{(4)},\ S_{8B}^{(4)}.$$

- Isolated matrices:

$$A_{8A}^{(0)},\ A_{8B}^{(0)},\ V_{8A}^{(0)},\ V_{8B}^{(0)}.$$

- Non-affine families:

$$T_8^{(1)},\ T_{8B}^{(3)}.$$

Affine: [4, 6, 7]; Isolated: [1, 2, 3]; Non-affine: [1, 2].

# Function $T_{8C}$

Consider a matrix-valued function $T_{8C}(a, b, c, d, e, f) =$

$$\begin{bmatrix} a & bd & d & ab & ad & bc & ac & bd \\ bd & -a & ab & -d & bc & -ad & bd & -ac \\ d & ab & bdef & -aef & acf & bdf & bce & -ade \\ ab & -d & -aef & -bdef & bdf & -acf & -ade & -bce \\ ac & bd & adf & bcf & -cdf & -abcf & -ac & -bcd \\ bd & -ac & bcf & -adf & -abcf & cdf & -bcd & ac \\ ad & bc & bde & -ace & -ac & -bcd & -abce & cde \\ bc & -ad & -ace & -bde & -bcd & ac & cde & abce \end{bmatrix}.$$

# Orthogonality Constraints of $T_{8C}(a, b, c, d, e, f)$

$$(T_{8C})(T_{8C})^{\dagger} =$$

$$\begin{bmatrix} 8 & -p_1/m_1 & p_3/m_2 & -p_2/m_2 & 0 & 0 & 0 & 0 \\ p_1/m_1 & 8 & p_2/m_2 & -p_3/m_2 & 0 & 0 & 0 & 0 \\ -p_2/m_1 & -p_3/m_1 & 8 & p_1/m_1 & 0 & 0 & 0 & 0 \\ p_3/m_1 & -p_2/m_1 & p_1/m_1 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & -p_1/m_1 & p_3/m_4 & -p_2/m_4 \\ 0 & 0 & 0 & 0 & p_1/m_1 & 8 & p_2/m_4 & p_3/m_4 \\ 0 & 0 & 0 & 0 & p_2/m_3 & p_3/m_3 & 8 & -p_1/m_1 \\ 0 & 0 & 0 & 0 & p_3/m_3 & p_2/m_3 & p_1/m_1 & 8 \end{bmatrix},$$

where $m_1, m_2, m_3, m_4$ are monomials and $p_1, p_2, p_3$ polynomials on $a, b, c, d, e, f$.

# Solution

We can solve $a, b, c$ with respect to $d, e, f$ from

$$\begin{cases} p_1 = a^2c - a^2b^2c + a^2c^2 - b^2c^2 + a^2d^2 - b^2d^2 + cd^2 - b^2cd^2 = 0 \\ p_2 = a^2bc + bcd^2 - a^2cde + b^2cde + abc^2f + abd^2f - acdef + ab^2cdef = 0 \\ p_3 = acd - ab^2cd + abc^2e + abd^2e + a^2cdf - b^2cdf + a^2bcef + bcd^2ef = 0 \end{cases}$$

and obtain a solution

$$d' = -\sqrt{cA_1}/\sqrt{B_1}, \quad e' = d'(A_3 + \sqrt{C})/B_3, \quad f' = (A_2 + \sqrt{C})/B_2,$$

where $A_1, A_2, A_3, B_1, B_2, B_2, C \in \mathbb{C}[a, b, c]$.

# Family $T_{8C}^{(3)}$

- The solutions $d'$, $e'$ and $f'$ are roots or multivariate palindromic polynomials so they get unimodular values exactly when $C(a,b,c)/(a^8b^8c^4) \leq 0$.
- $T_{8C}^{(3)} := \mathcal{S}_C(D)$, where

$$\mathcal{S}_C(a,b,c) = T_{8C}(a,b,c,d',e',f'),$$

and

$$D = \{(a,b,c) \in \mathbb{T}^3 \mid C/(a^8b^8c^4) \leq 0,\ B_1, B_2, B_3 \neq 0\},$$

is a 3-parameter family of complex Hadamard matrices.

# Family $T_{8D}^{(3)}$

Similar considerations result in a family arising from a matrix-valued function $T_{8D}(a, b, c, d, e, f) =$

$$\begin{bmatrix} 1 & -c & c & -e & -c & e & -e & ce \\ 1 & c & -e & c & -e & c & -e & -ce \\ c & -1 & c & e & -c & -e & ce & -e \\ c & 1 & e & c & e & c & ce & e \\ b & f & ad & ab & df & bf & abd & adf \\ b & -f & ab & ad & -bf & -df & abd & -adf \\ f & b & -ad & ab & -df & bf & -adf & -abd \\ f & -b & ab & -ad & -bf & df & -adf & abd \end{bmatrix}.$$

# Family $T_{8E}^{(3)}$

Family $T_{8E}^{(3)}$ can be obtained from $T_{8E}(a,b,c,d,e,f) =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & d & e & -f & df & ef & -de & def \\ 1 & a & -b & f & -af & bf & ab & abf \\ 1 & -d & b & c & cd & -bc & bd & bcd \\ 1 & -a & -e & -c & -ac & -ce & -ae & ace \\ 1 & -ad & be & -1 & ad & -be & -abde & abde \\ 1 & ad & -1 & -cf & -acdf & cf & -ad & acdf \\ 1 & -1 & -be & cf & -cf & -bcef & be & bcef \end{bmatrix}.$$

# Family $T_{8F}^{(3)}$

Family $T_{8F}^{(3)}$ can be obtained from $T_{8F}(a, b, c, d, e, f, g, h) =$

$$\begin{bmatrix} 1 & ab & ef & abef & cd & dh & cg & gh \\ ab & 1 & -abef & -ef & dh & cd & -gh & -cg \\ ef & -abef & -1 & ab & -cg & gh & cd & -dh \\ abef & -ef & ab & -1 & -gh & cg & -dh & cd \\ be & -ae & bf & -af & 1 & -ch & dg & -cdgh \\ ae & -be & -af & bf & ch & -1 & -cdgh & dg \\ bf & af & -be & -ae & -dg & -cdgh & 1 & ch \\ af & bf & ae & be & -cdgh & -dg & -ch & -1 \end{bmatrix}.$$

# Algorithm to determine if $H \in H_n^{(k)}$

Consider $H \in \mathbb{H}(n)$ and $H_n^{(k)} = f(D)$, where $f$ is iterativly invertable function.

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1. Find all mutually permutation inequivalent dephased matrices $H'$ equivalent to $H$.
2. For each $H' = (h'_{i,j})$, perform a backtracking search to find parameters $x' = (x'_1, \ldots, x'_k)$, such that $\langle f_{i,j}(x') \rangle = \langle h'_{i,j} \rangle$, by evaluating suitable inverted entries of $f$ with elements from $\{h'_{i,j}\}$.

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3. For each pair $(H', x')$, determine if $f(x'_1, \dots, x'_k)$ and $H'$ are permutation equivalent.

# Inequivalence

## Theorem

*Let $H_n^{(k)}$ be a $k-$parameter family of complex Hadamard matrices of order 8 from the list*

$$\begin{gathered} F_8^{(5)},\ D_{8A}^{(5)},\ D_{8B}^{(5)}, S_{8A}^{(4)},\ S_{8B}^{(4)}, \\ T_{8B}^{(3)},\ T_{8C}^{(3)},\ T_{8D}^{(3)},\ (T_{8D}^{(3)})^T, T_{8E}^{(3)}, (T_{8E}^{(3)})^T,\ T_{8F}^{(3)}. \end{gathered} \tag{0.1}$$

*Let $f : D \to \mathbb{H}(8)$ to be the function defining $H_8^{(k)}$ as given in [8]. Consider $H' = f(e^{2i}, e^{3i}, \dots, e^{(k+1)i}) \in \mathbb{H}(8)$. The only family from the list (0.1) to which $H'$ belongs is $H_8^{(k)}$ itself.*

# Uncategorized Butson matrices or order 8

The BH$(8,6)$ matrix

$$B_1 = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^3 & \omega^3 & \omega^3 & \omega^3 \\ \omega^0 & \omega^1 & \omega^3 & \omega^4 & \omega^0 & \omega^2 & \omega^3 & \omega^5 \\ \omega^0 & \omega^2 & \omega^5 & \omega^3 & \omega^4 & \omega^5 & \omega^2 & \omega^1 \\ \omega^0 & \omega^3 & \omega^2 & \omega^5 & \omega^2 & \omega^5 & \omega^1 & \omega^4 \\ \omega^0 & \omega^3 & \omega^4 & \omega^1 & \omega^5 & \omega^2 & \omega^0 & \omega^3 \\ \omega^0 & \omega^4 & \omega^1 & \omega^3 & \omega^3 & \omega^2 & \omega^5 & \omega^0 \\ \omega^0 & \omega^5 & \omega^3 & \omega^2 & \omega^1 & \omega^5 & \omega^4 & \omega^2 \end{bmatrix}$$

does not belong in any of the known families.

Classifications of Butson matrices: [5].

# Summary

- Overview of Complex Hadamard matrices of order 8.
- Novel families $T_{8C}^{(3)}$, $T_{8D}^{(3)}$, $(T_{8D}^{(3)})^T$, $T_{8E}^{(3)}$, $(T_{8E}^{(3)})^T$, $T_{8F}^{(3)}$.
- Algorithm to decide if $H \in H_n^{(k)}$.
- There exists BH$(8, q)$ matrices that do not belong in any of the known families.

# Thank you!

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