On higher-dimensional Hadamard matrices and designs

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Joint work with Mario Osvin Pavčević, Lucija Relić and Kristijan Tabak

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What is a higher-dimensional Hadamard matrix?

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An n-dimensional matrix of order v with $\{-1,1\}$-entries

$$H : \{1,\ldots,v\}^n \to \{-1,1\}$$

- is Hadamard, if all $(n-1)$-dim. parallel sections are orthogonal:

$$\sum_{1\le i_1,\ldots,\widehat{i_j},\ldots,i_n\le v} H(i_1,\ldots,a,\ldots,i_n)H(i_1,\ldots,b,\ldots,i_n) = v^{n-1}\delta_{ab}$$

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J. Hammer, J. Seberry, *Higher-dimensional orthogonal designs and Hadamard matrices II*, Proc. Ninth Manitoba Conference on Numerical Mathematics and Computing, pp. 23–29, Congress. Numer. XXVII, Utilitas Math., Winnipeg, 1980.

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$$\left[\mathrm{Prod}\left(H, H^{\tau^{n-1}},\ldots,H^{\tau^2},H^{\tau}\right)\right]_{i_1,\ldots,i_n} = \begin{cases} v, & \text{if } i_1=\ldots=i_n,\\ 0, & \text{otherwise.}\end{cases}$$

E. K. Gnang, Y. Filmus, *On the spectra of hypermatrix direct sum and Kronecker products constructions*, Linear Algebra Appl. **519** (2017), 238–277.

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A Hadamard matrix

$n = 2, \ \ v = 4$

$$\begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix}$$

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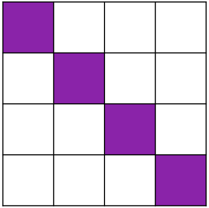
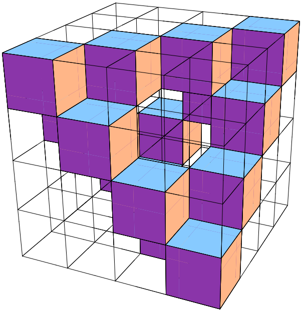


A three-dimensional proper Hadamard matrix

$n = 3, \quad v = 4$



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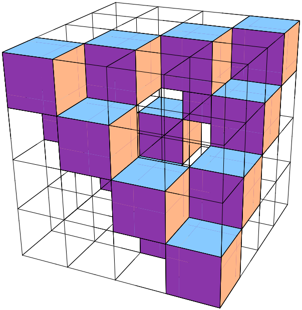

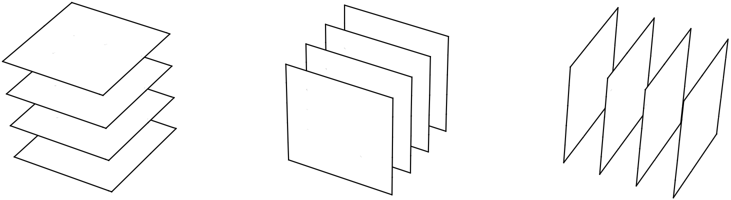

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Theorem ("Product construction")

Let $h : \{1, \ldots, v\}^2 \to \{-1, 1\}$ be a 2-dimensional Hadamard matrix of order v. Then

$$H(i_1, \ldots, i_n) = \prod_{1 \leq j < k \leq n} h(i_j, i_k)$$

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Proof. All 2-dimensional slices of H are equivalent to h.

Other types of higher-dimensional designs

Other types of combinatorial designs: symmetric block designs (SBIBDs), orthogonal designs, (generalized) weighing matrices...

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Chapter 11: *Origins of cocyclic development*

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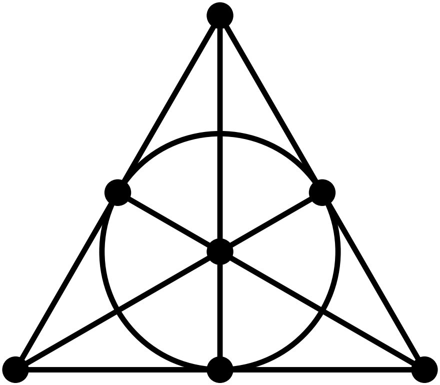
Chapter 11: *Origins of cocyclic development*

V. Krčadinac, M. O. Pavčević, K. Tabak, *Cubes of symmetric designs*, Ars Math. Contemp. **25** (2025), no. 1, #P1.10.

The Fano plane



The Fano plane

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

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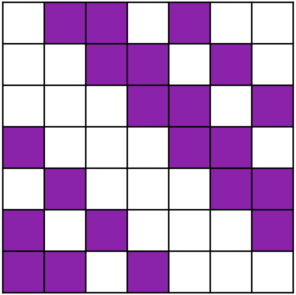
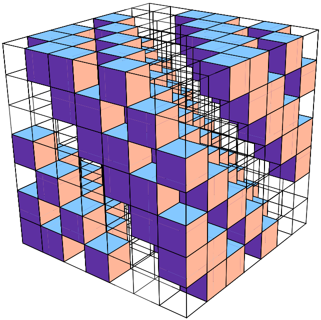


A three-dimensional Fano cube



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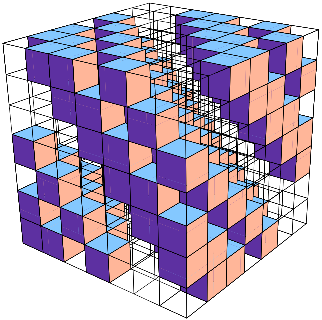

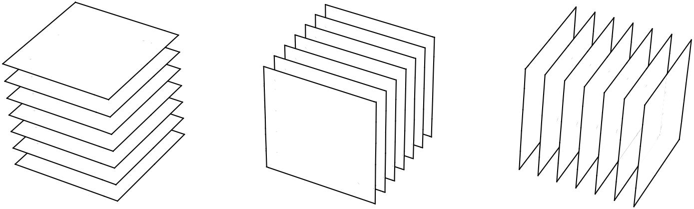

Cubes of symmetric designs

Theorem ("Difference cubes")

If D is a (v, k, λ) difference set in $G = \{g_1, \ldots, g_v\}$, then

$$A(i_1, \ldots, i_n) = [\, g_{i_1} + \ldots + g_{i_n} \in D \,]$$

is an n-dimensional cube of symmetric (v, k, λ) designs.

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Questions:

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Questions:

1. Are there cubes of symmetric designs not coming from this theorem? ("non-difference cubes")
2. Are there cubes of symmetric designs with inequivalent slices?

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Theorem (V.K., M. O. Pavčević, K. Tabak: "Group cubes")

If $\{D_1, \ldots, D_v\}$ are the blocks of a symmetric (v, k, λ) design, and each D_i is a (v, k, λ) difference set in $G = \{g_1, \ldots, g_v\}$, then

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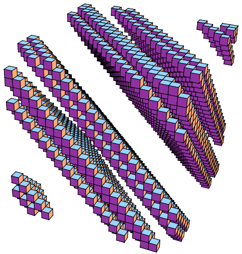
A 3-cube of $(21, 5, 1)$ designs
(projective planes of order 4)



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$$G=\langle a,b \mid a^3=b^7=1,\ ba=ab^2\rangle$$

$$D_1=\{1,a,b,b^3,a^2b^2\}$$
$$D_2=\{a^2b^6,b^6,a^2b^3,a^2b^4,a\}$$
$$D_3=\{1,a^2,ab,b^2,b^6\}$$
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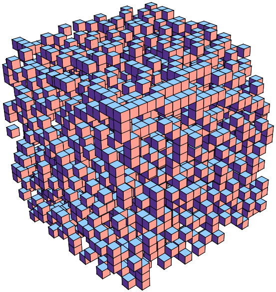

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For every $m \geq 2$ and $n \geq 3$, there are n-cubes of symmetric

$$(4^m,\ 2^{m-1}(2^m-1),\ 2^{m-1}(2^{m-1}-1))$$

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There are three $(16, 6, 2)$ designs:

$$|\operatorname{Aut}(\mathcal{D}_1)| = 11520, \quad |\operatorname{Aut}(\mathcal{D}_2)| = 768, \quad |\operatorname{Aut}(\mathcal{D}_3)| = 384$$

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Red design, Green design, Blue design

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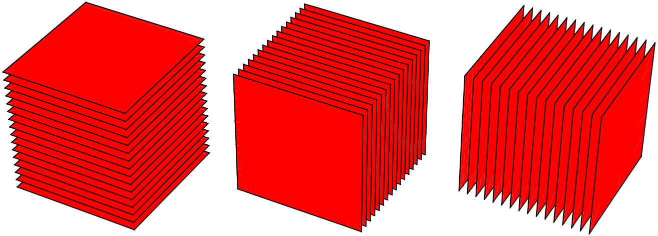

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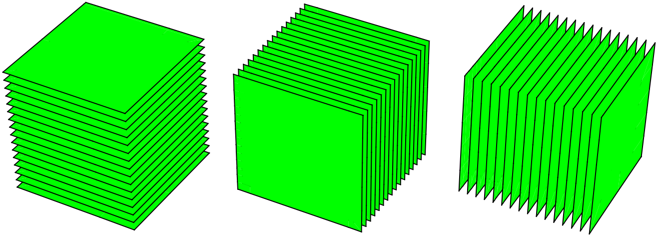

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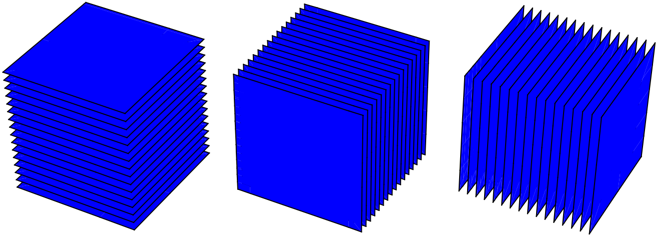

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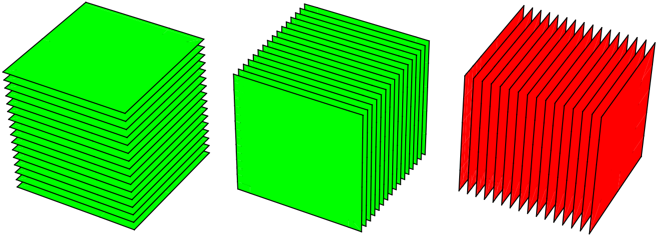

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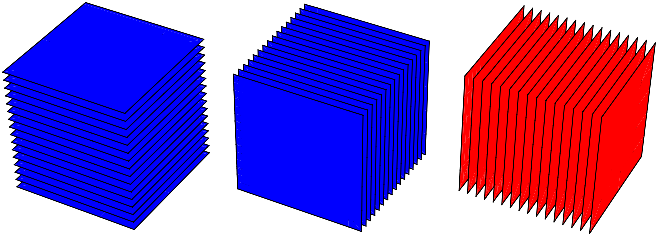

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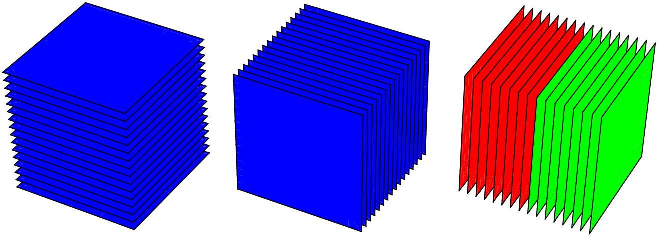

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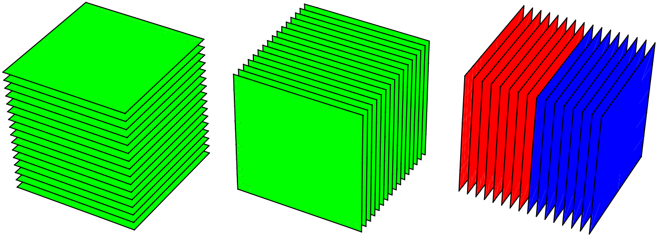

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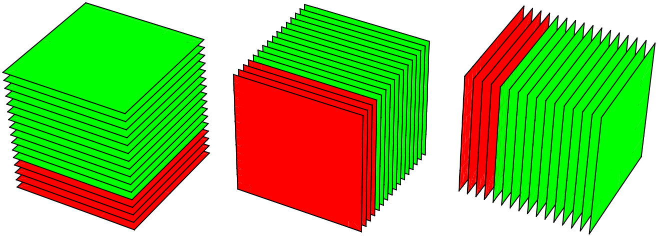

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Proposition.

Up to equivalence, the set $\mathcal{C}^3(16,6,2)$ contains exactly 27 difference cubes and 946 non-difference group cubes. Furthermore, it contains at least 1423 inequivalent non-group cubes.

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designs that are group cubes, but not difference cubes.

The parameters are of Menon type: $(4u^2, 2u^2-u, u^2-u)$

By exchanging $0 \to -1$, the cubes are transformed to n-dimensional proper Hadamard matrices with inequivalent slices!

Cubes of symmetric designs

Open questions:

1. There are exactly 78 symmetric $(25, 9, 3)$ designs, but no difference sets. Are there cubes of $(25, 9, 3)$ designs of dimension $n \geq 3$?

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3. **Is there a product construction for cubes of symmetric designs?**
4. Hadamard matrices coming from Menon designs are of square orders. Are there n-dimensional proper Hadamard matrices with inequivalent slices of non-square orders?

A forgotten success story: Room squares

T. G. Room, *A new type of magic square*, Math. Gaz. **39** (1955), 307.

Thomas Gerald Room

From Wikipedia, the free encyclopedia

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Let S be a set of $v+1$ elements, say $S = \{\infty, 1, 2, \ldots, v\}$.

A Room square of order v is a $v \times v$ matrix M such that:

- the entries of M are empty or 2-element subsets of S
- each 2-subset of S appears once in M
- elements of S appear once in every row and column of M

Example.

$v = 7$

$\infty 1$			26		57	34
45	$\infty 2$			37		16
27	56	$\infty 3$			14	
	13	67	$\infty 4$			25
36		24	17	$\infty 5$		
	47		35	12	$\infty 6$	
		15		46	23	$\infty 7$

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Theorem.

Room squares of order v exists if and only if v is odd and $v \neq 3, 5$.

Proof: 1955–1973.

Higher-dimensional Room cubes

A Room cube is an n-dimensional matrix of order v with entries that are empty or 2-subsets of $S = \{\infty, 1, 2, \ldots, v\}$ such that every 2-dimensional **projection** is a Room square.

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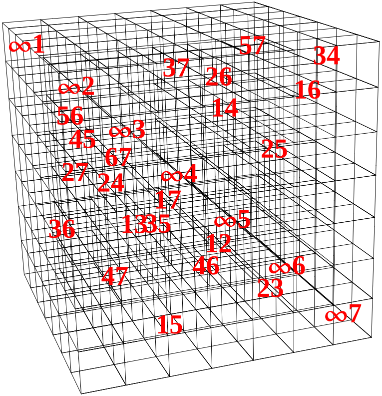

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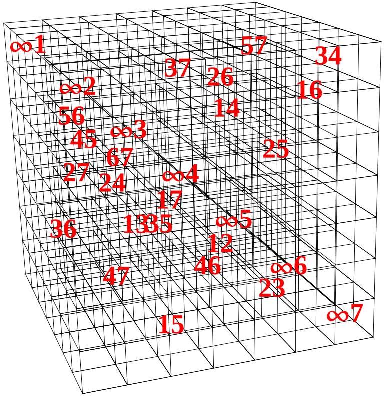

Front view:

∞1	56	24		37		
	∞2	67	35		14	
		∞3	17	46		25
36			∞4	12	57	
	47			∞5	23	16
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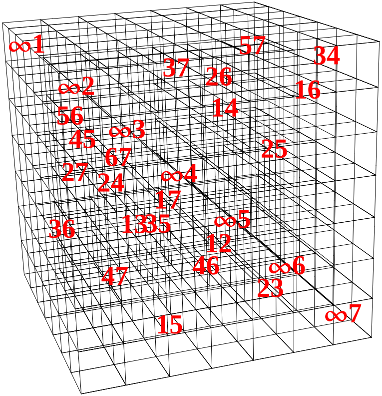

Top view:

∞1			36		27	45
56	∞2			47		13
24	67	∞3			15	
	35	17	∞4			26
37		46	12	∞5		
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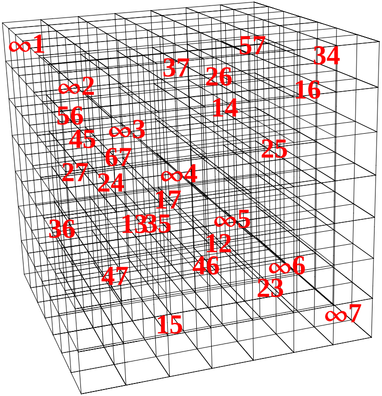

Side view:

26	34		57			∞1
45		16			∞2	37
	27			∞3	14	56
13			∞4	25	67	
		∞5	36	17		24
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∞7	15	23		46		

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A Room square of order v is equivalent to:

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Conjecture (W. D. Wallis): $\nu(v) \leq \frac{1}{2}(v-1)$

The End

Thanks for your attention!