Mutually Unbiased Bases
for
Continuous Variables

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Outline

- continuous variables: $\mathbb{C}^d$ with $d \to \infty$
  - mutually unbiased (MU) bases
  - complex Hadamard matrices
- known sets of MU bases for continuous variables
- open problems & conjectures

Mutually unbiased bases in $\mathbb{C}^d$

- **consider two ON bases**

$$\mathcal{B}_b = \{|v_b\rangle, v = 1 \dots d\}\,, \qquad \mathcal{B}_{b'} = \{|v'_{b'}\rangle, v' = 1 \dots d\}$$

- $\mathcal{B}_b$ **and** $\mathcal{B}_{b'}$ **are mutually unbiased iff**

$$|\langle v'_{b'}|v_b\rangle|^2 = \begin{cases} \delta_{v'v} & \text{for} \quad b = b' \\ \frac{1}{d} & \text{for} \quad b \neq b' \end{cases}$$

- **complex Hadamard matrices**
  - $H_{v'v} = \langle v'_{b'}|v_b\rangle$ are unitary and "flat"

MU bases in prime dimensions $d = p$

- **Heisenberg-Weyl group**

$$ZX = \omega XZ\,, \qquad \omega^d = 1$$

  - **phase and (cyclic) shift operators**

$$Z|k\rangle = \omega^k |k\rangle\,, \qquad X|k\rangle = |k+1\rangle\,, \qquad k = 1 \dots d$$

- **complete set of $(d+1)$ pairwise MU bases**
  - **given by eigenstates of**

$$X\,, Z\,, XZ\,, XZ^2\,, \dots XZ^{d-1}$$

  striations of $d \times d$ phase space grid

Example: $d = 3$ with $\omega = e^{2\pi i/3}$ (or $\omega^3 = 1$)

complete set of **four** MU bases

$$\mathbb{I}_3, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega & 1 & \omega^2 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix}$$

three complex Hadamard matrices

$$\frac{1}{\sqrt{3}} \omega^{vk+bk^2}, \ldots \qquad k, v, b = 0, 1, 2$$

equivalences: permutations of rows & columns, rephasing columns, global unitaries

Sets of MU bases in composite dimensions $d \neq p^k$

- plausible generalizations of known constructions **fail**
- **long-standing open problem**
  - **Do complete sets exist for** $d = 6, 10, 12, 15, \ldots$**?**
  - survey: McNulty & Weigert 2024 - arXiv:2410.23997v1
- modifications: real/p-adic/quaternionic MU bases etc.
  - here: $\mathbb{C}^d \to L_2(\mathbb{R})$

Continuous variables (I)

quantum particle with a single degree of freedom

- state space $L_2(\mathbb{R})$

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \in L_2(\mathbb{R}) \qquad \text{if} \qquad \langle\psi|\psi\rangle = \sum_{n=0}^{\infty} |c_n|^2 = 1$$

- canonical commutation relations

$$i[\hat{p}, \hat{q}] = \hbar$$

- **no** normalizable flat vectors with non-zero overlap for $d \to \infty$**:**

$$\frac{1}{\sqrt{d}}(1, e^{is_1}, e^{is_2}, \ldots, e^{is_{d-1}})^T$$

- **Are we stuck?**

Continuous variables (II)

orthonormal bases with continuous indeces for $L_2(\mathbb{R})$

- generalized states $|q\rangle \notin L_2(\mathbb{R})$ s.t.

$$\hat{q}|q\rangle = q|q\rangle,\ q \in \mathbb{R}, \quad \langle q'|q\rangle = \delta(q'-q), \quad \int_{\mathbb{R}} dq\, |q\rangle\langle q| = \mathbb{I}$$

  - same for momentum $\hat{p}$
  - rigged Hilbert spaces, theory of distributions

- expansion

$$|p\rangle = \mathbb{I}|p\rangle = \int_{\mathbb{R}} dq\, |q\rangle\langle q|p\rangle = \int_{\mathbb{R}} dq\, \frac{e^{-iqp/\hbar}}{\sqrt{2\pi\hbar}}|q\rangle$$

- standard Fourier transform!

Continuous complex Hadamard matrices

- matrix elements $H_{pq} \in \mathbb{C}$
- continuous indices $p, q \in \mathbb{R}$

$$|H_{pq}| = \text{const.} \quad \text{and} \quad \int_{\mathbb{R}} dp\, H^{\dagger}_{pq'} H_{pq} = \delta(q' - q)$$

- general form:

$$H_{pq} = \frac{1}{\sqrt{2\pi\hbar}} e^{ih(p,q)}$$

- example: Fourier "matrix" → orthonormal set of vectors

$$H_{pq} = \frac{e^{-ipq/\hbar}}{\sqrt{2\pi\hbar}} : \qquad \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} dp\, e^{-ip(q-q')/\hbar} = \delta(q' - q)$$

Continuous mutually unbiased bases

consider **two** generalized **bases**

$$\{|s\rangle,\, s \in \mathbb{R}\}\,, \qquad \{|t\rangle,\, t \in \mathbb{R}\}$$

- orthonormality

$$\langle s'|s\rangle = \delta(s' - s) \quad \text{and} \quad \langle t'|t\rangle = \delta(t' - t)$$

- **mutually unbiased** (Wilkinson&SW2008)

$$|\langle s|t\rangle| = k > 0 \qquad (k \quad \text{not unique!})$$

- position & momentum

$$\langle q|p\rangle = \frac{e^{-iqp/\hbar}}{\sqrt{2\pi\hbar}}\,, \quad \text{so} \quad k = \frac{1}{\sqrt{2\pi\hbar}}$$

Triple of continuous MU bases

Heisenberg-Weyl group:

$$e^{is\hat{p}/\hbar}\, e^{it\hat{q}/\hbar} = e^{-ist/\hbar}\, e^{it\hat{q}/\hbar}\, e^{is\hat{p}/\hbar}$$

- phase-space translations with generators $\hat{q}$ and $\hat{p}$

$$e^{it\hat{q}/\hbar}|q\rangle = e^{itq/\hbar}|q\rangle\,, \qquad e^{is\hat{p}/\hbar}|q\rangle = |q+s\rangle\,, \qquad q, s.t \in \mathbb{R}$$

three continuous pairwise MU bases

- take generalized eigenstates $|q\rangle$, $|p\rangle$, $|r\rangle$ of $\hat{q}, \hat{p}, \hat{r} \equiv -(\hat{p}+\hat{q})$

$$[\hat{p},\hat{q}] = [\hat{q},\hat{r}] = [\hat{r},\hat{p}] = \frac{\hbar}{i}$$

  associated with lines in phase space

- **two continuous CHMs** (position representation)

$$\left\{ \delta(q'-q), \frac{1}{\sqrt{2\pi\hbar}} e^{-ipq/\hbar}, \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}(rq-q^2/2)} \right\}$$

Three continuous MU bases

phase space translation operators $(\mathbf{a} = (Q, P)^T, \hat{x} = (\hat{q}, \hat{p})^T)$:

$$\hat{T}(\mathbf{a}) = e^{i(P\hat{q} - Q\hat{p})/\hbar} = e^{i\mathbf{a}\cdot J\cdot\hat{\mathbf{x}}/\hbar}$$

where

$$\boldsymbol{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

HW-triple of MU bases

$$\{|p\rangle, p \in \mathbb{R}\}, \quad \{|q\rangle, q \in \mathbb{R}\}, \quad \{|r\rangle, r \in \mathbb{R}\}$$

- associated with **phase-space directions**

$$\boldsymbol{e}_p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{e}_q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_r = -\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

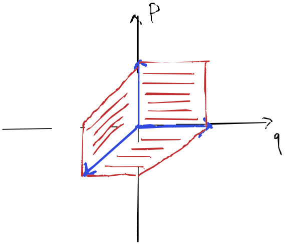
MU-ness: $|\langle p|q\rangle|^2 = |\langle q|r\rangle|^2 = |\langle r|q\rangle|^2 = 1/2\pi\hbar \qquad \Longrightarrow$

$$|\boldsymbol{e}_p^t \cdot \boldsymbol{J} \cdot \boldsymbol{e}_q| = |\boldsymbol{e}_q^t \cdot \boldsymbol{J} \cdot \boldsymbol{e}_r| = |\boldsymbol{e}_r^t \cdot \boldsymbol{J} \cdot \boldsymbol{e}_p| = 1$$

Phase space visualization

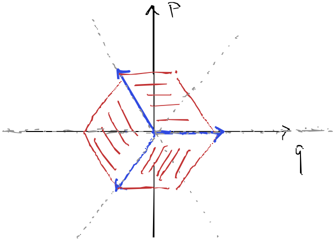
asymmetric triple

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



symmetric triple

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}$$



Equivalences for sets of continuous MU bases

sets of MU bases / Hadamard matrices may be **equivalent**

$$\{\mathcal{B}_s, \mathcal{B}_t\} \sim \{\mathcal{B}_t, \mathcal{B}_s\} \sim \{\mathcal{B}_t, \mathcal{B}_u\} \sim \ldots$$

equivalences from **symmetry transformations** of overlaps

$$|\langle q'_{b'}|q_b\rangle|^2 = \frac{1}{2\pi\hbar}, \qquad \text{for all } q, q', b \neq b'$$

- global unitaries: $|q_b\rangle \to U|q_b\rangle$
- permutations of bases: $\mathcal{B}_b \to \mathcal{B}_{\sigma(b)}$
- **permutations within bases:** $|q_b\rangle \to |\pi(q)_b\rangle$
- rephasing vectors: $|q_b\rangle \to e^{if(q_b)}|q_b\rangle$
- overall complex conjugation: $|q_b\rangle \to K|q_b\rangle$

$\Longrightarrow$ **dephased sets of continuous Hadamard matrices**: one real row/column

- generalized **Haagerup criterion?**

Continuous MU bases for $N = 1$

- three MU bases $\{|q\rangle\}$, $\{|p\rangle\}$, $\{|r\rangle\}$ of HW-type (WW2008)
- inequivalent triples of continuous MU bases?
  - assume a (generalized) state $|\psi\rangle$ MU to both $\{|q\rangle\}$ and $\{|p\rangle\}$

$$|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} dq \, e^{if(q)} |q\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} dq \, e^{ig(p)} |p\rangle$$

  - need to solve an integral equation (f, g not quadratic!)

$$e^{ig(p)} = \frac{1}{\sqrt{2\pi\hbar}} \int_{\mathbb{R}} dq \, e^{if(q)} e^{-ipq/\hbar}$$

  - construct a continuous ON basis $\{|\psi_s\rangle, s \in \mathbb{R}\}$ from $|\psi\rangle$
- quadruples? (no known upper limit!)

Continuous MU bases for $N = 2$

one quintuple of **product** MU bases (WW2008):

- consider five product vectors:

$$\boldsymbol{e}_p \otimes \boldsymbol{e}_p, \quad \boldsymbol{e}_q \otimes \boldsymbol{e}_q$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1-\phi \end{pmatrix} \begin{pmatrix} 1 \\ \phi \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2-\phi \end{pmatrix} \begin{pmatrix} 1 \\ 1+\phi \end{pmatrix}$$

where $\phi^2 = \phi + 1$

Continuous MU bases for $N = 2$

four one-parameter families of quintuples of product MU bases

- consider family of five product vectors:

$$\boldsymbol{e}_p \otimes \boldsymbol{e}_p, \quad \boldsymbol{e}_q \otimes \boldsymbol{e}_q$$

$$\begin{pmatrix} 1 \\ \lambda \end{pmatrix} \begin{pmatrix} 1 \\ -1/\lambda \end{pmatrix}, \quad \begin{pmatrix} -\phi \\ \lambda \end{pmatrix} \begin{pmatrix} 1-\phi \\ 1/\lambda \end{pmatrix}, \quad \begin{pmatrix} 1-\phi \\ \lambda \end{pmatrix} \begin{pmatrix} -\phi \\ 1/\lambda \end{pmatrix}$$

  $\lambda \in \mathbb{R}$ (Beales&SW2022)

- equivalences**?**
- **no** sextuples of product form
- **no known other types** of MU bases
- **no known entangled** MU bases
- **no known upper limit** on # of MU bases

Summary and open questions

- **continuous complex Hadamard matrices and MU bases**
  - $d \to \infty$ requires **generalized** states and orthogonality relations on $L_2(\mathbb{R})$
  - **analogous** HW group constructions
  - limited set of results for $N = 1, 2$
  - classification seems difficult
- **open questions**
  - **Haagerup criterion** for equivalence of continuous CHMs**?**
  - show **uniqueness** of continuous basis MU to $\{|q\rangle\}$ and $\{|p\rangle\}$**?**
  - continuous MU bases **not of HW-type?**
  - $N = 2$: **entangled** continuous MU bases**?**
  - **equi-angular lines** and continuous MU bases**?**