Intrinsic volumes of the quantum state space and mutually unbiased bases

M. Weiner and Sz. Zsombor

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$$\forall j \neq k : |\langle u_j, u_k \rangle|^2 = \begin{cases} 0, & \text{if } \left\lfloor \frac{j}{d} \right\rfloor = \left\lfloor \frac{k}{d} \right\rfloor \\ \frac{1}{d}, & \text{otherwise} \end{cases}$$

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However, in no dimension it excludes the existence of a compl. MUB syst.
(or the existence of a SIC-POVM)

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Let E be a d^2-1 dim. Euclidean space. Given some vectors $a_1, \dots, a_n \in E$ s.t.: $\|a_1\| = \dots = \|a_n\| = \sqrt{1-\frac{1}{d}}$ and for $j \neq k$: $\langle a_j, a_k \rangle \geq -\frac{1}{d}$, decide if $\exists\, u_1 \dots u_n \in \mathbb{C}^d$ unit vectors

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* where $A_j = |u_j\rangle\langle u_j| - \frac{1}{d} I$

The role of the set of density matrices

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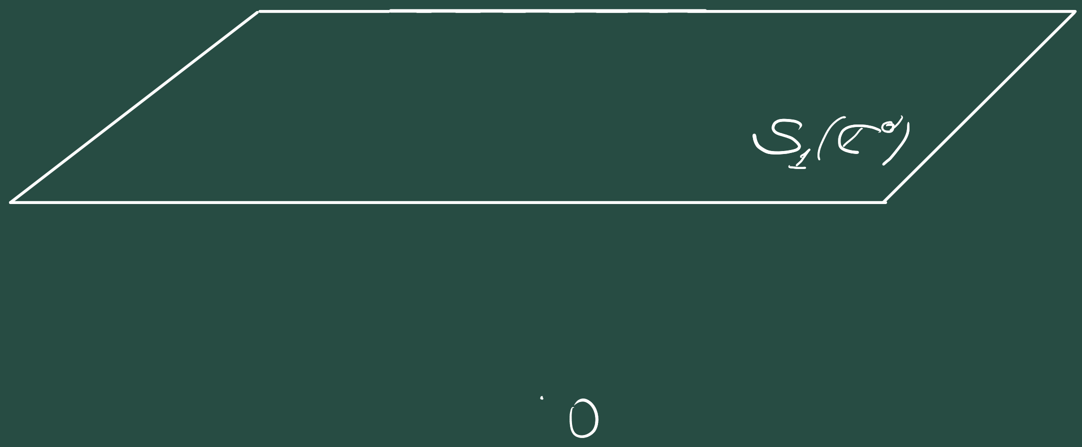

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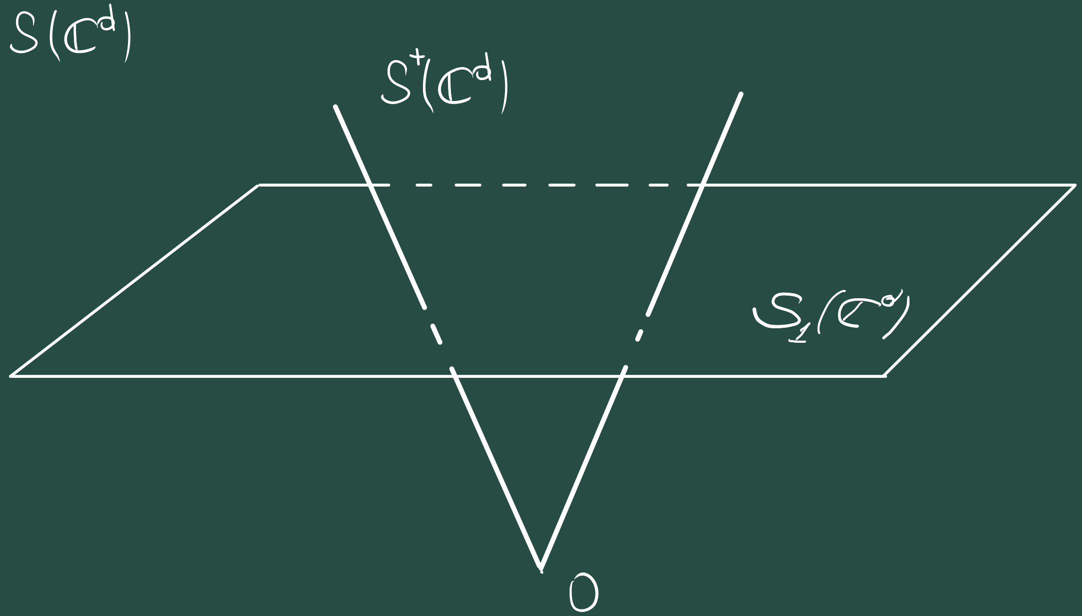

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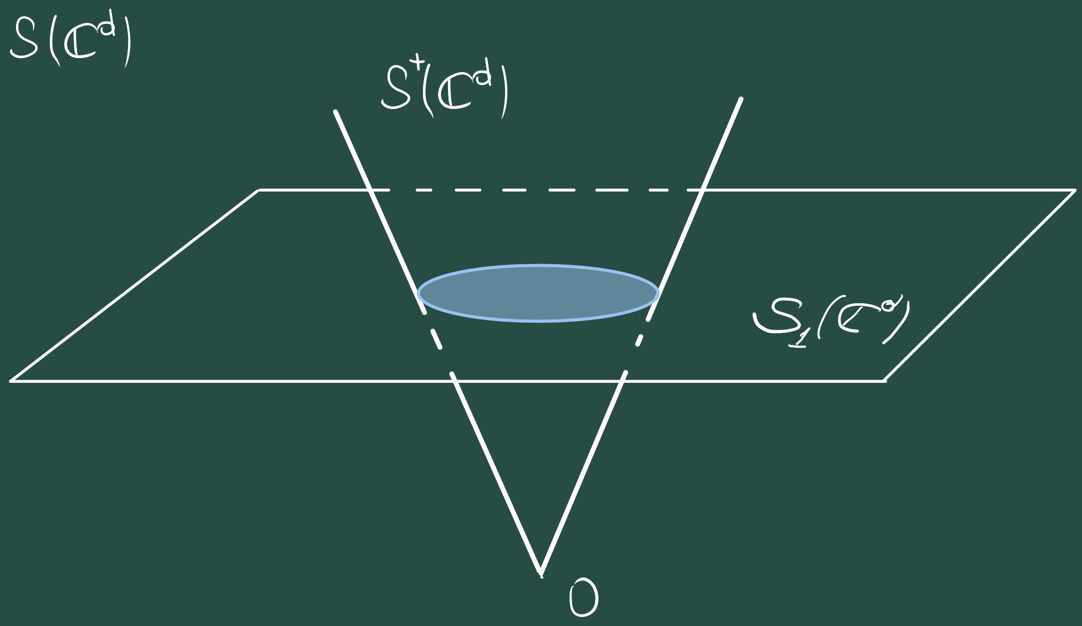

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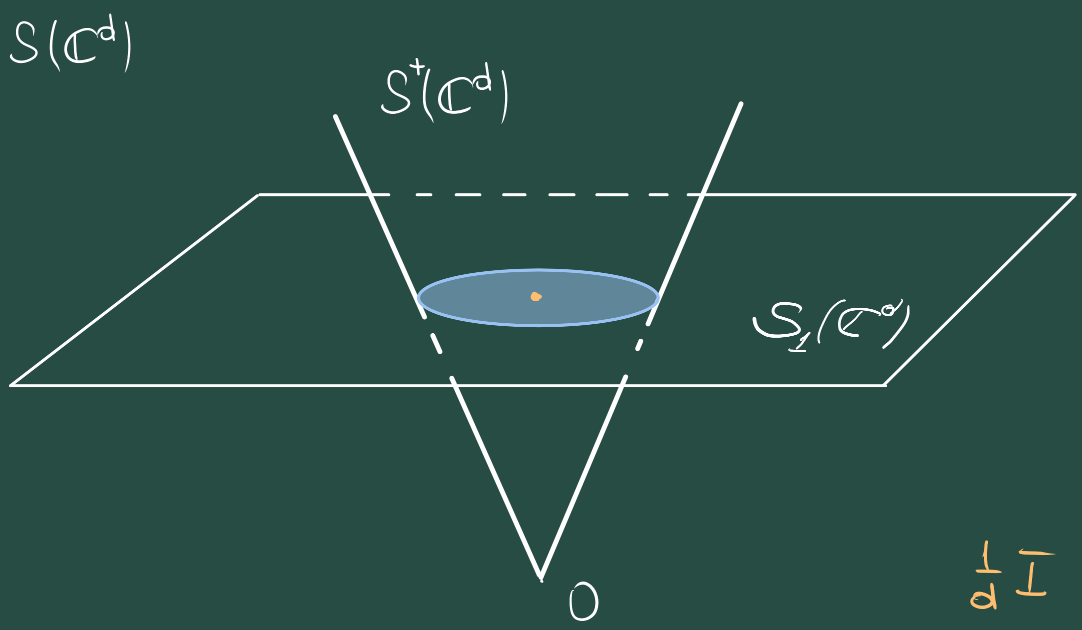

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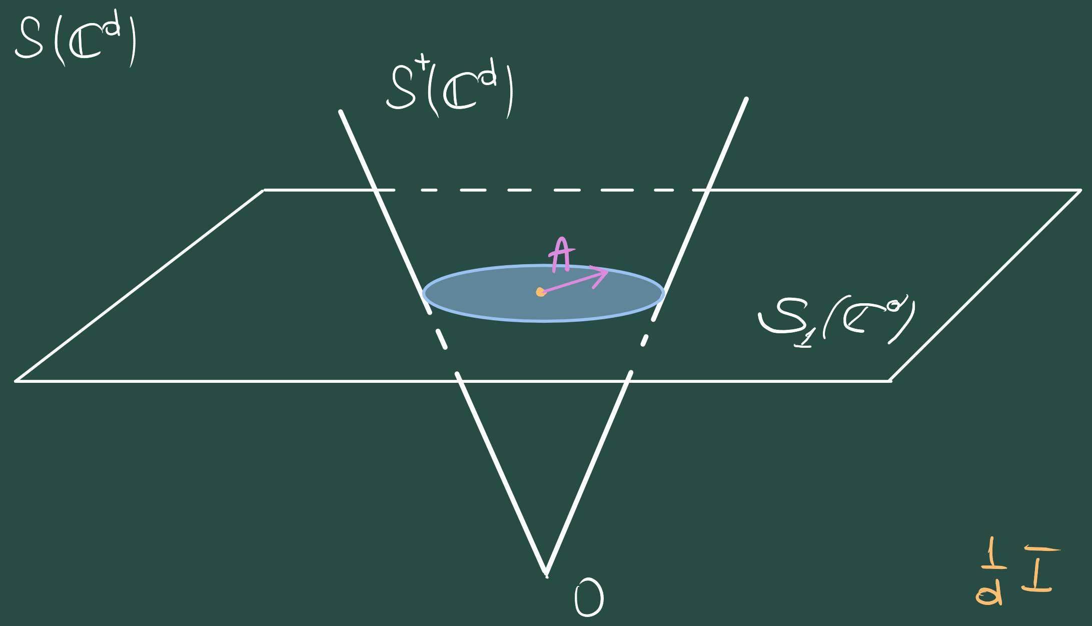

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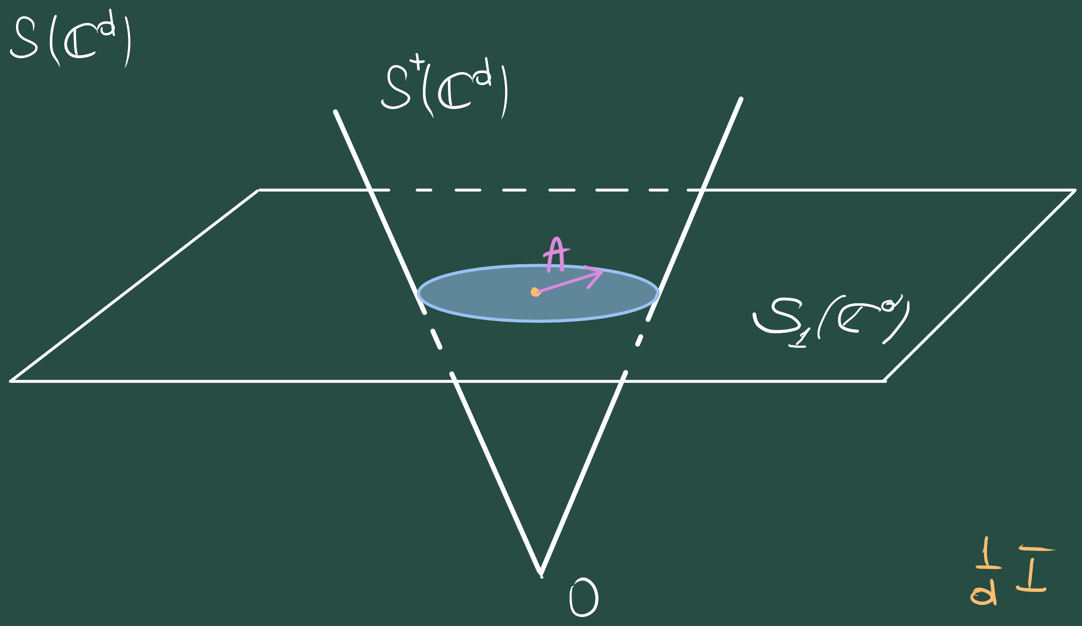

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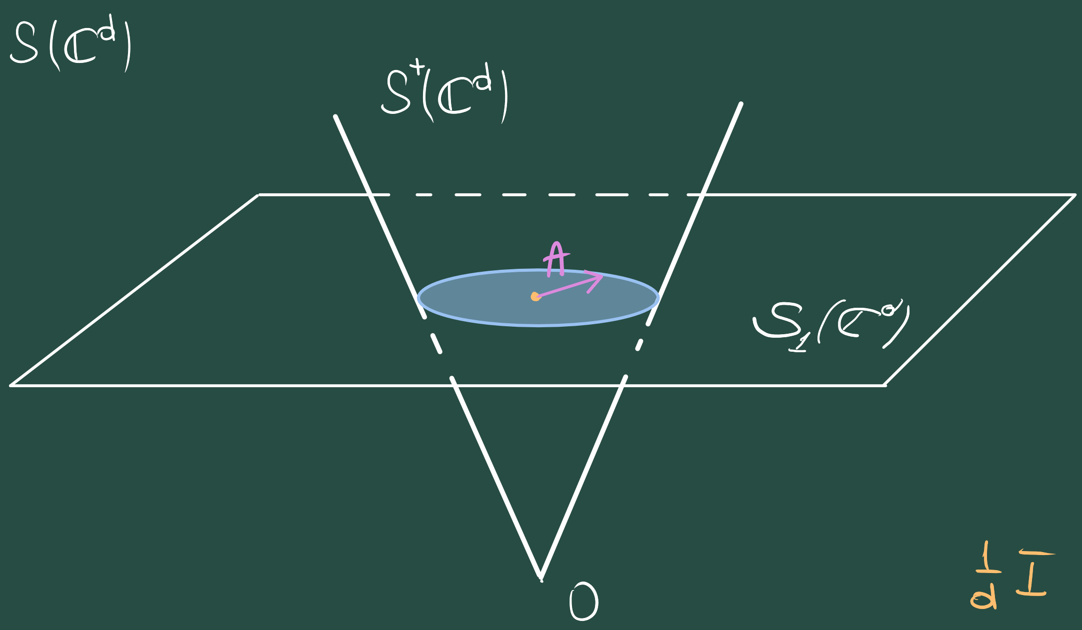

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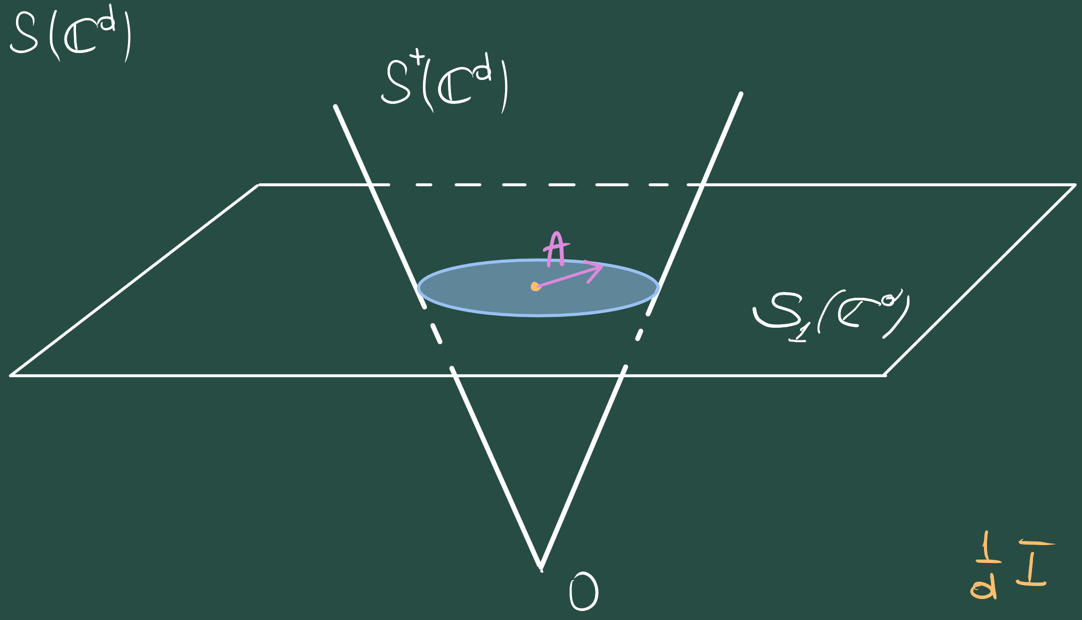

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$\Longleftrightarrow$

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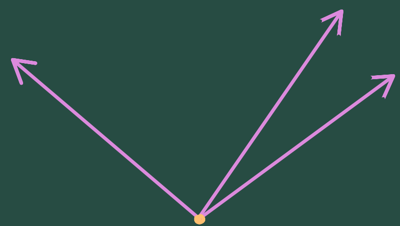

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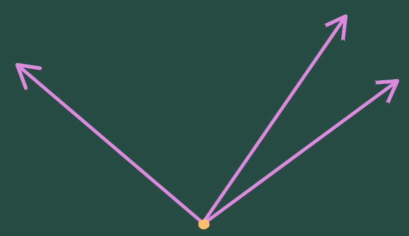

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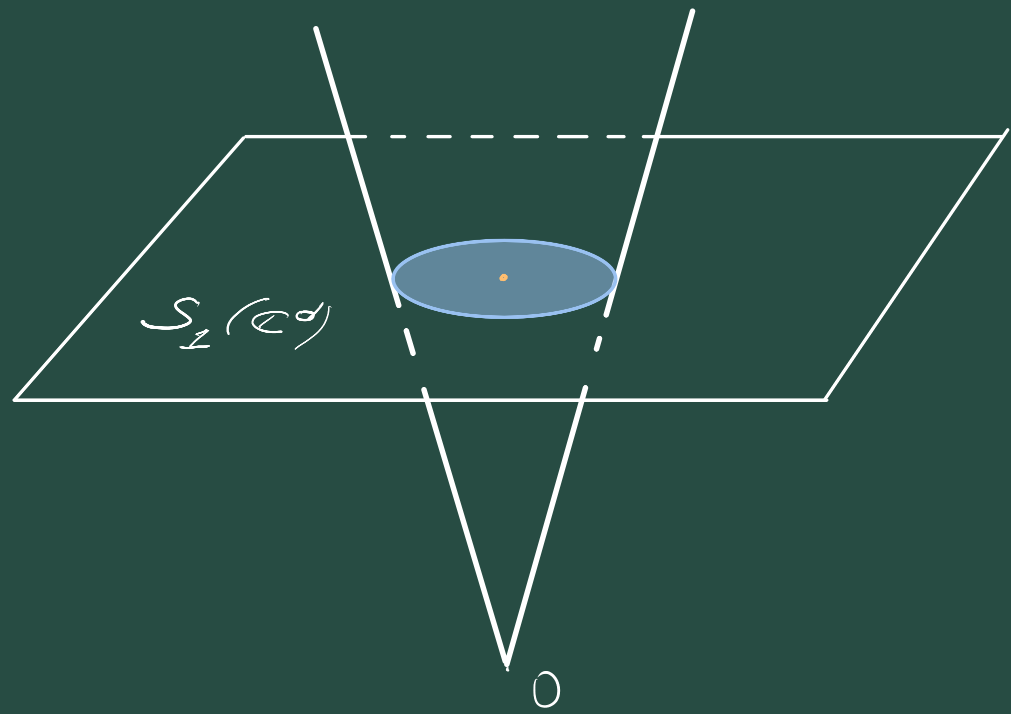

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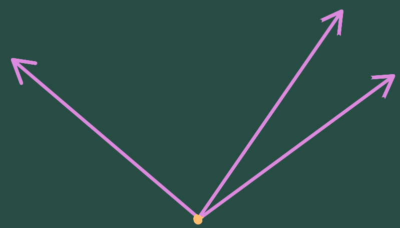

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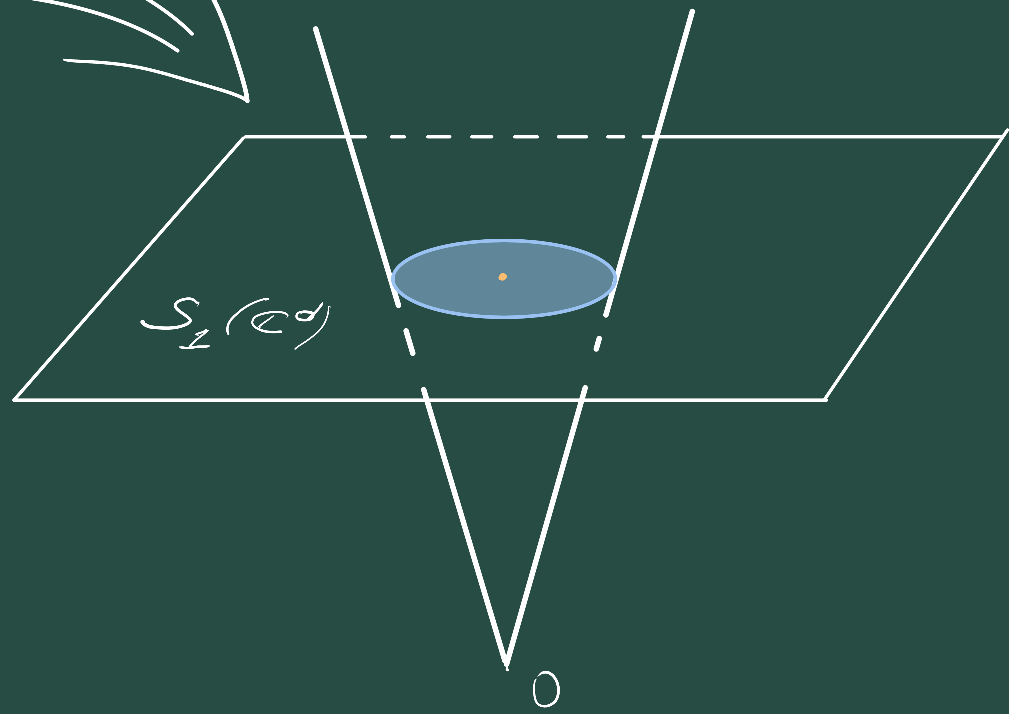

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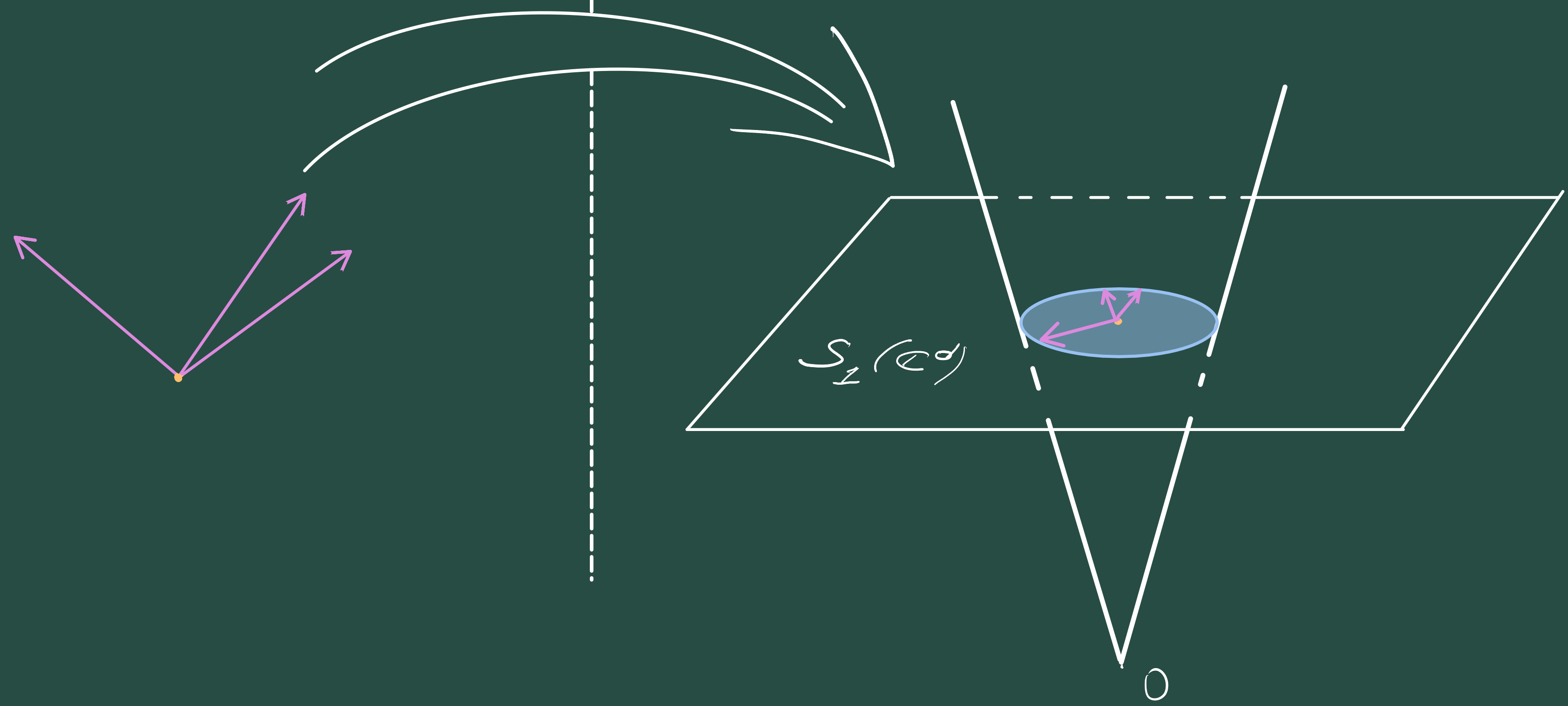

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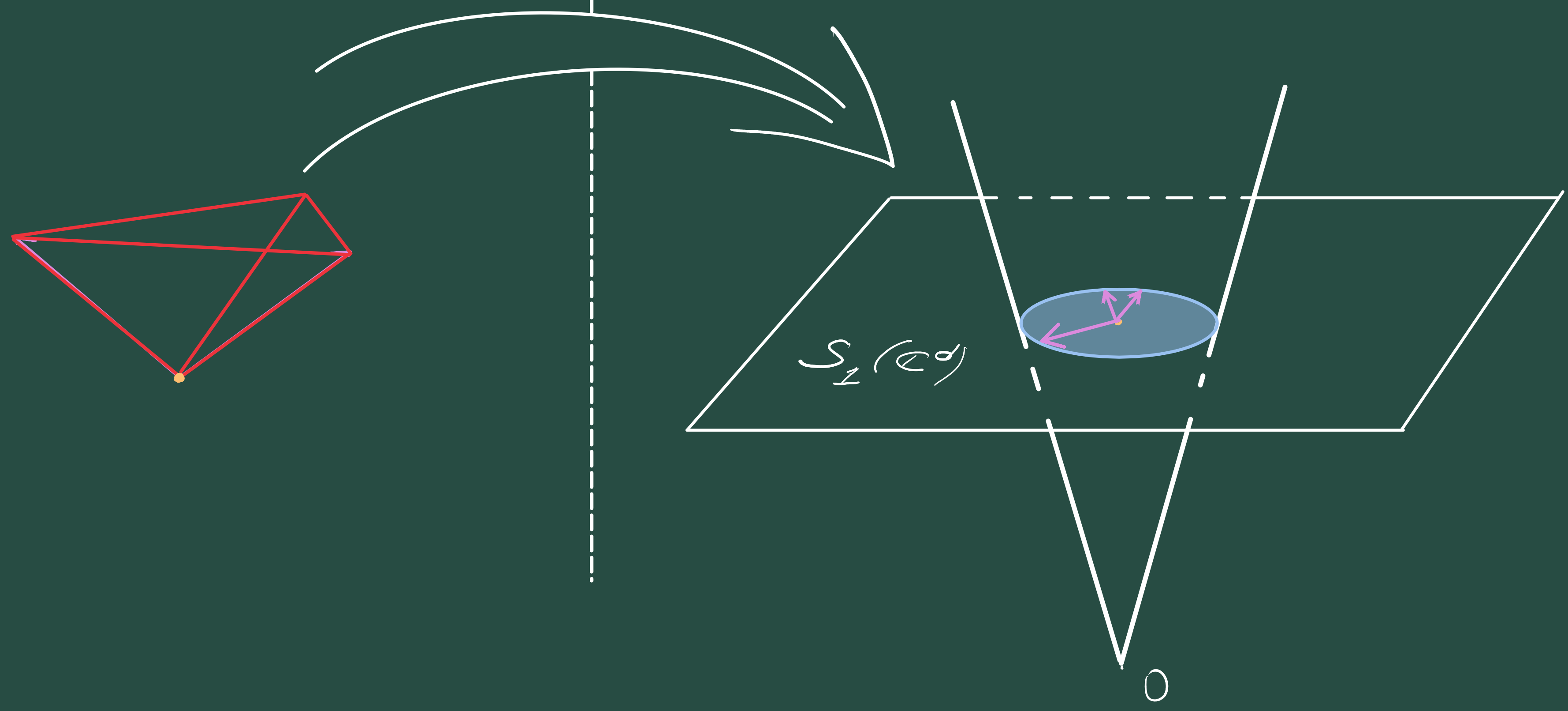

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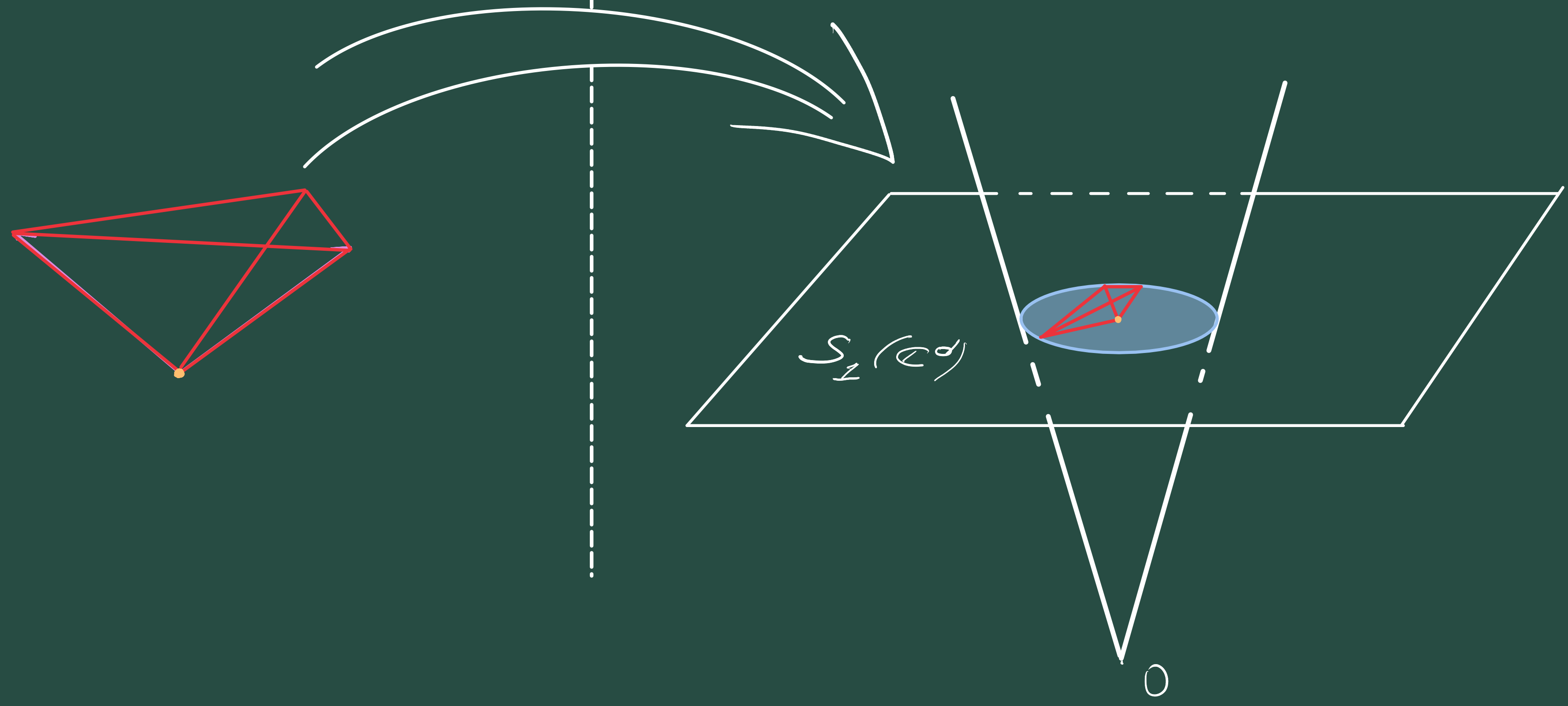

The polytope in the MUB case

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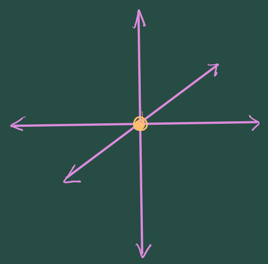

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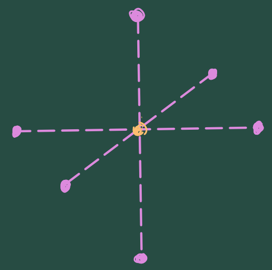

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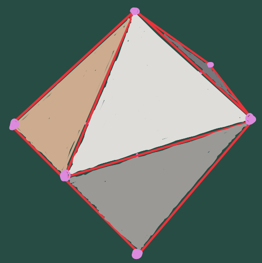

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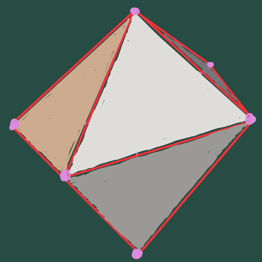

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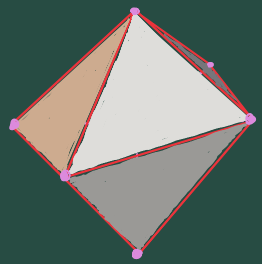

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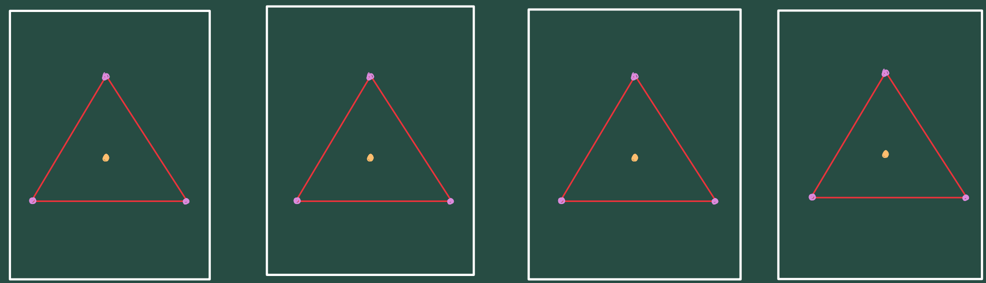

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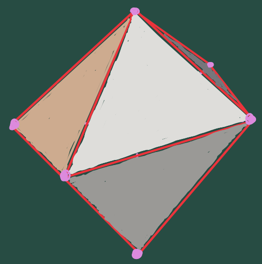

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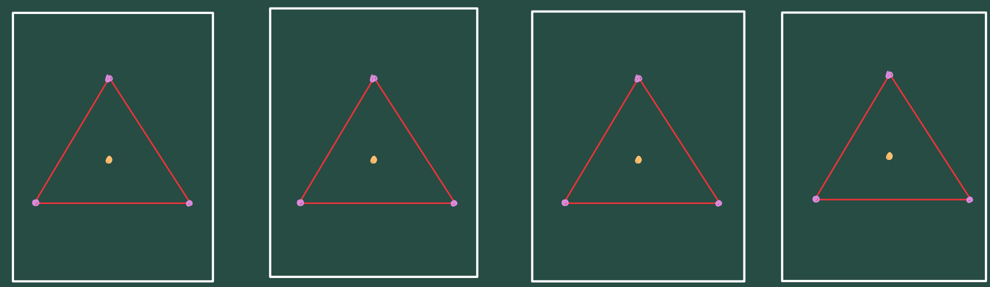

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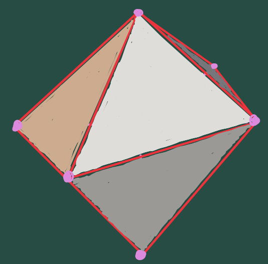

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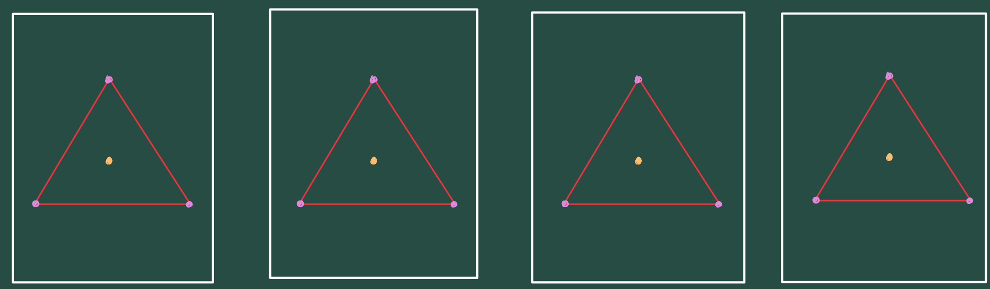

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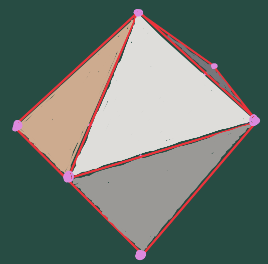

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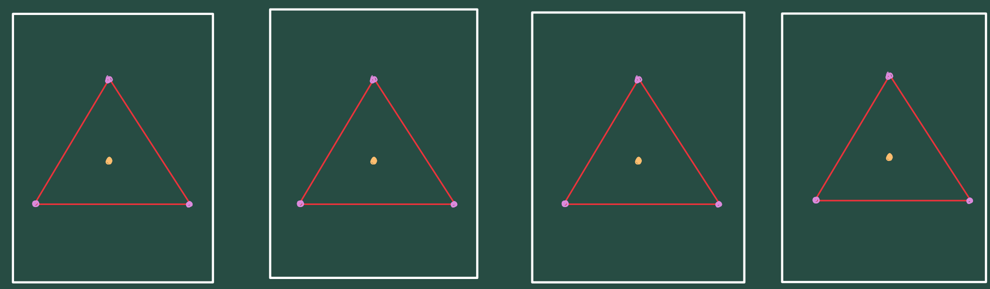

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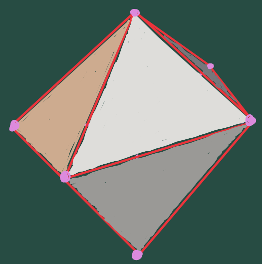

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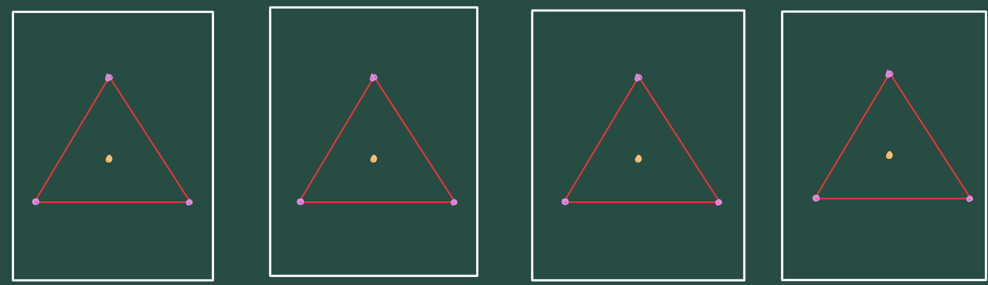

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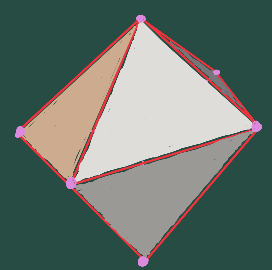

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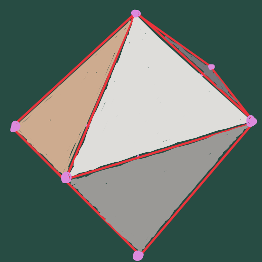

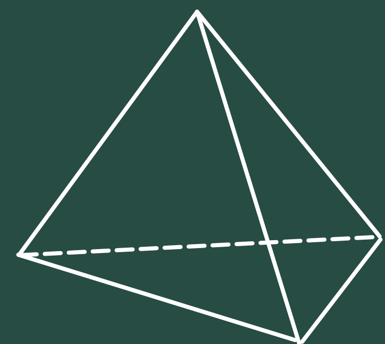

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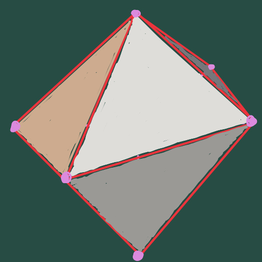
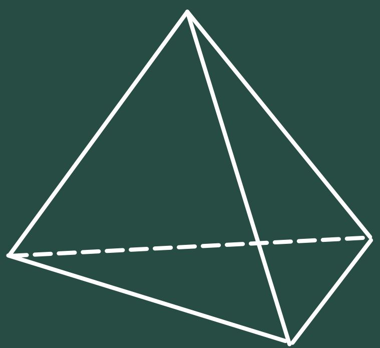
 $\hookrightarrow$ 

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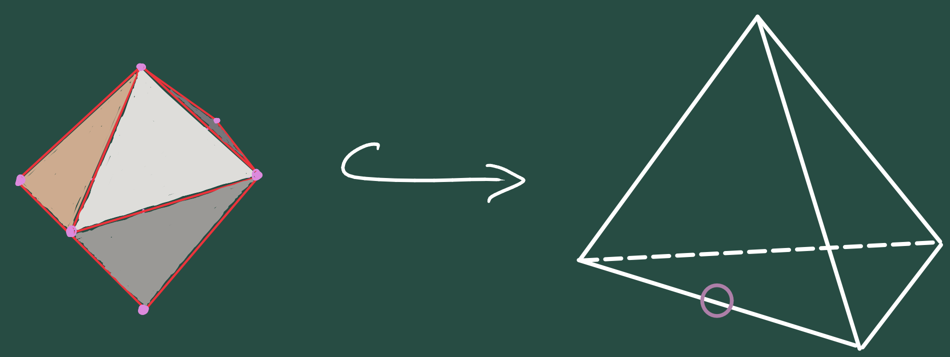

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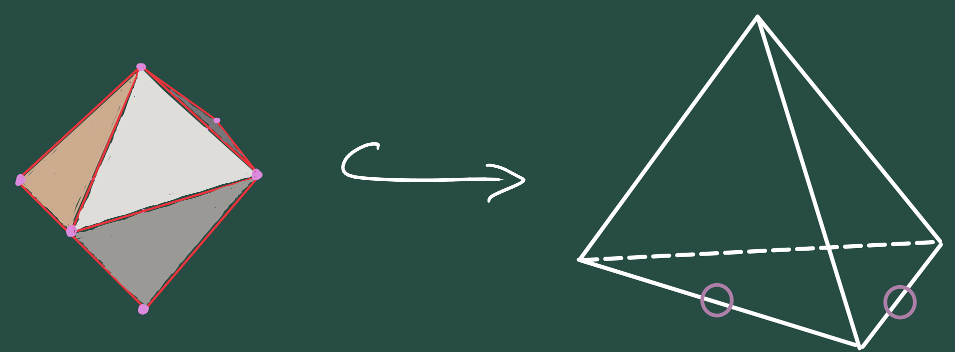

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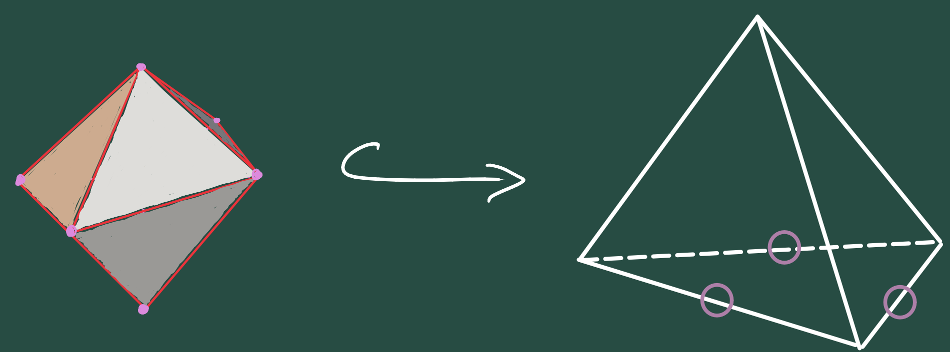

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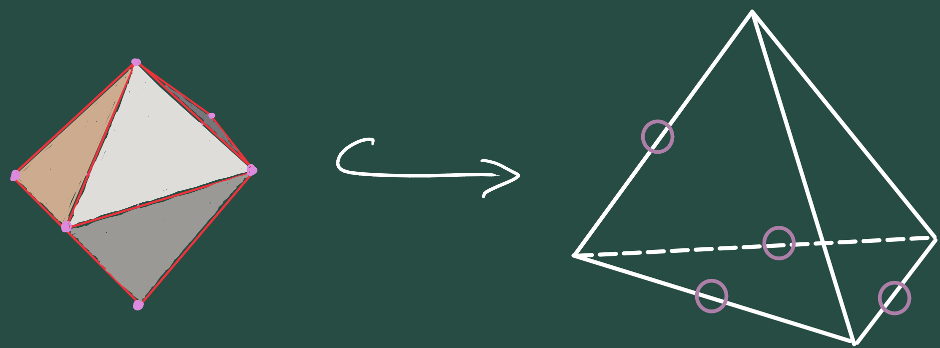

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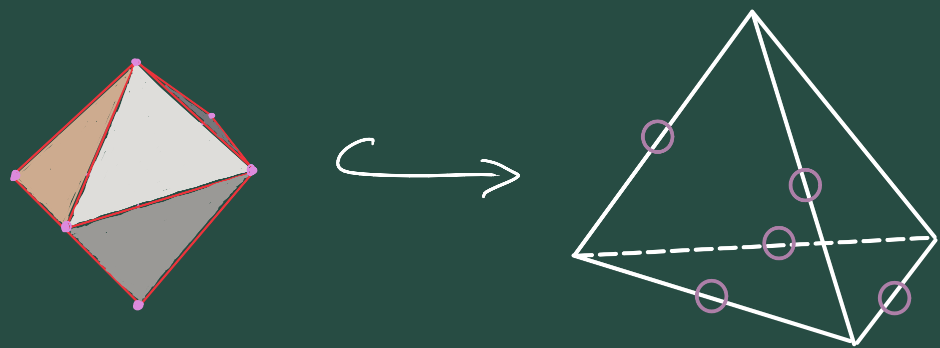

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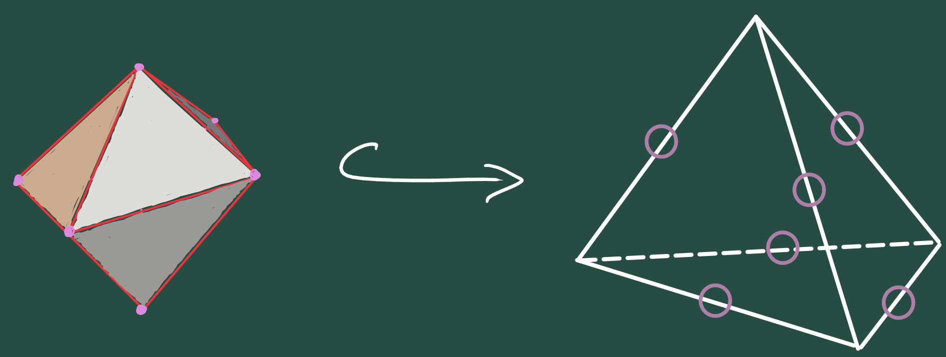

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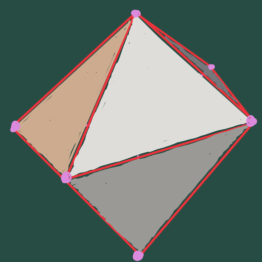
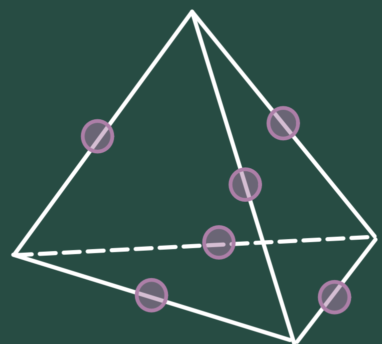
 $\hookrightarrow$ 

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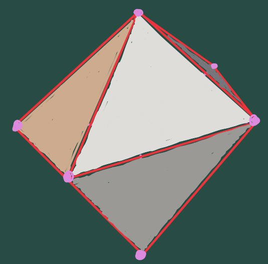

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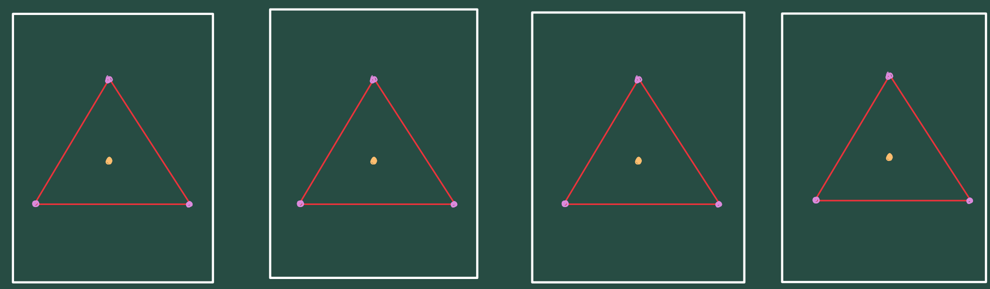

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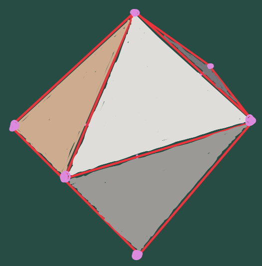

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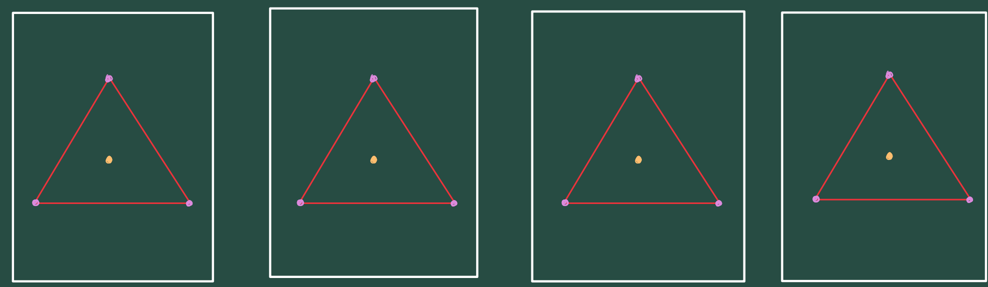

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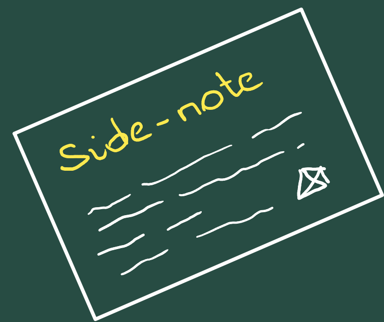

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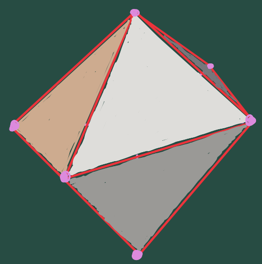

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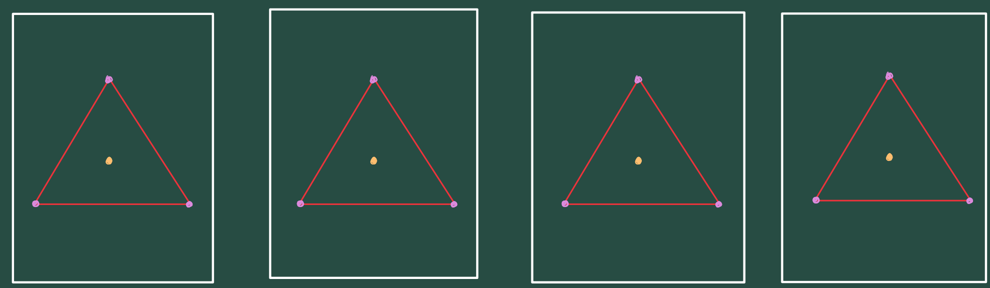

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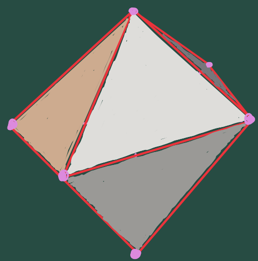

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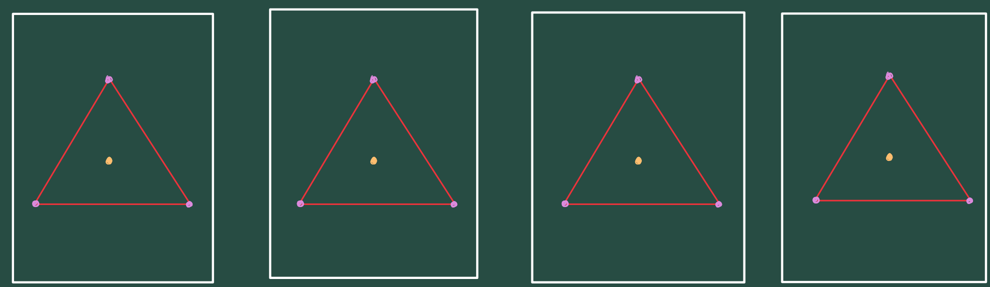

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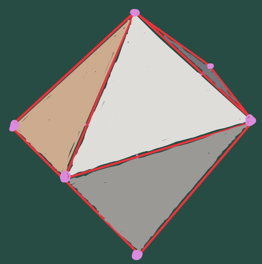

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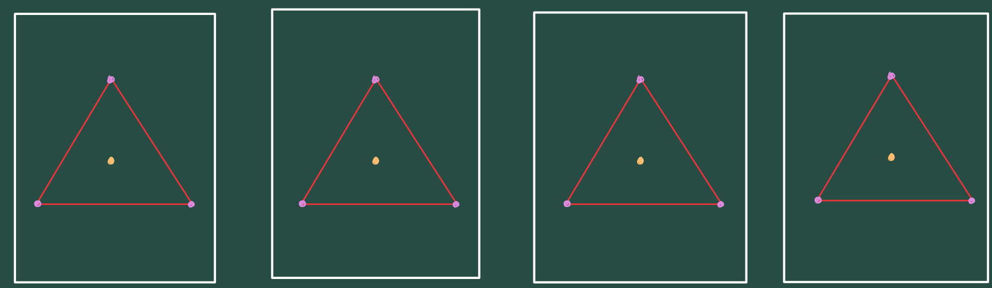

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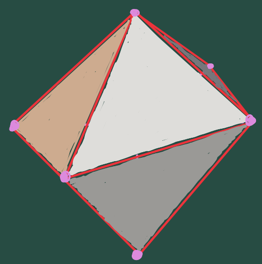

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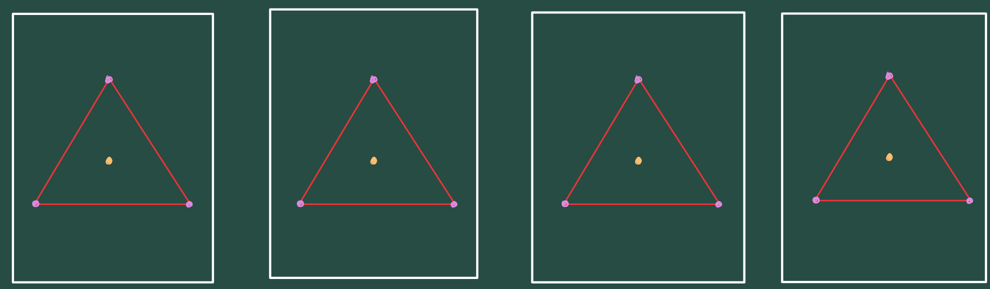

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Question : what further quantities we can look for ?

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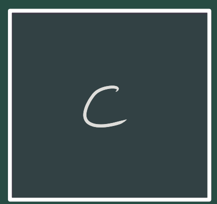

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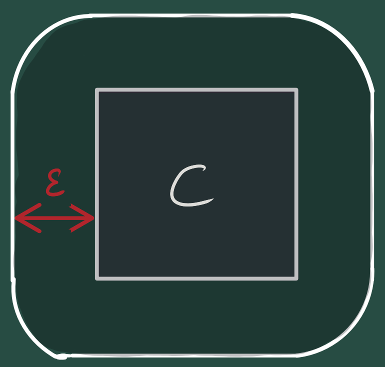

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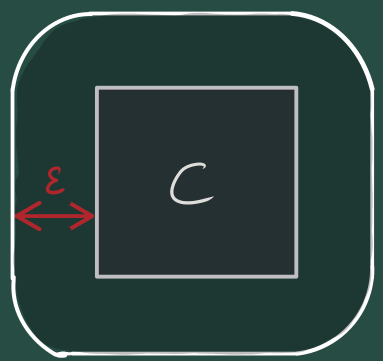

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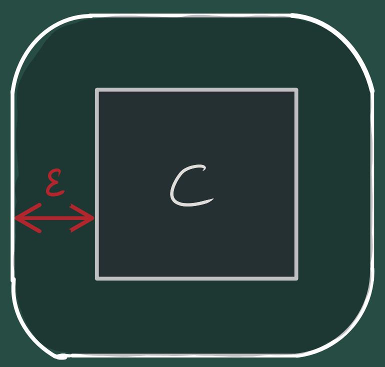

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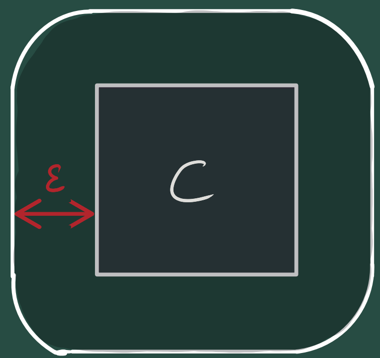


$$a_0 = Vol_D(C)$$

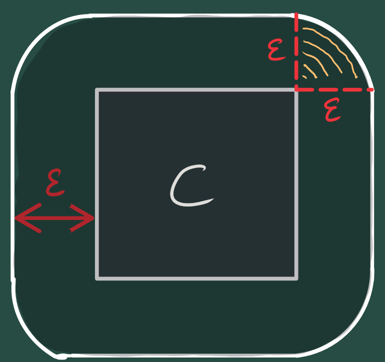
$$a_1 = A(C) = Vol_{D-1}(C)$$

Intrinsic volumes

$\mathcal{E}$: a Euclidean space of dim. D (in our case $D = d^2 - 1$)

$C \subset \mathcal{E}$ a compact convex set, $C_\varepsilon := \{ q \in \mathcal{E} \mid \min_{x \in C} \| q - x \| \leq \varepsilon \}$

$$\Rightarrow \quad \mathrm{Vol}(C_\varepsilon) = a_0 + a_1 \varepsilon + a_2 \varepsilon^2 + \ldots + a_D \varepsilon^D$$



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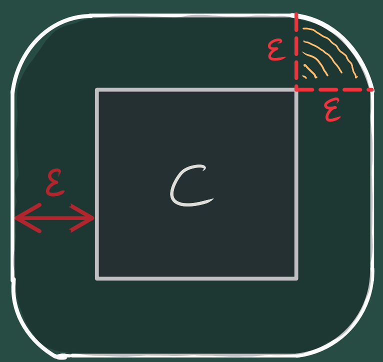

$$a_0 = \mathrm{Vol}_D(C)$$

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A factor that does not depend on C

$$\boxed{\hookrightarrow \mathrm{Vol}_k(C) \equiv a_{D-k} \cdot \chi_{k,D}}$$

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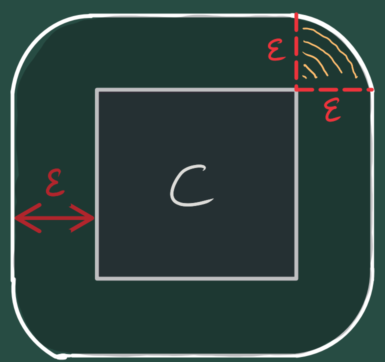

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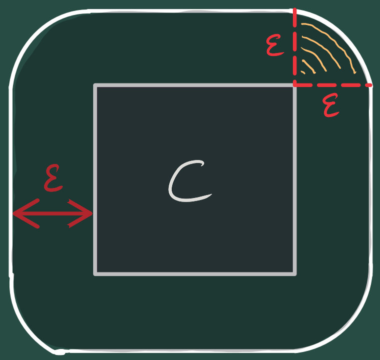
A factor that does not depend on C

Vol$_k$ is monotone :

$C, \tilde{C}$ are convex and $C \subset \tilde{C} \quad \Rightarrow \quad \mathrm{Vol}_k(C) \leq \mathrm{Vol}_k(\tilde{C})$

Can we compute Vol_{κ} of a polytope ?

Can we compute Vol_ε of a polytope ?



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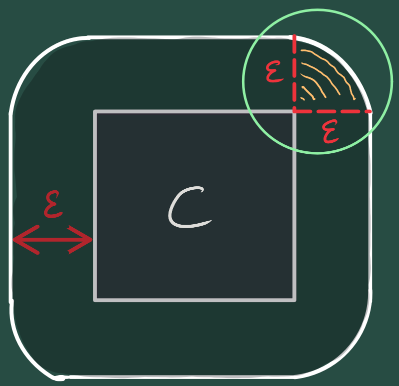

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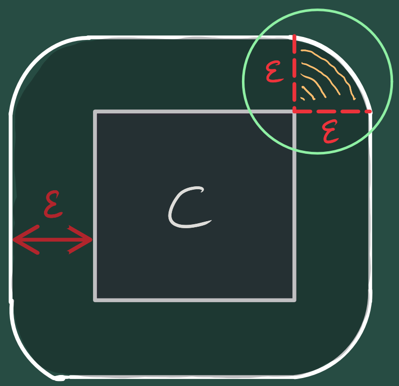

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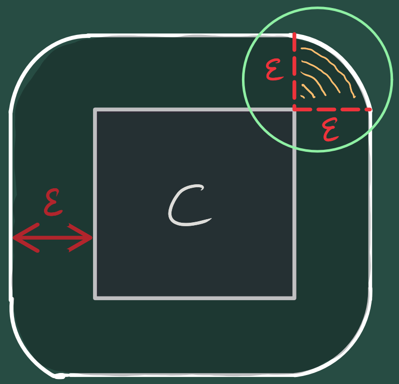
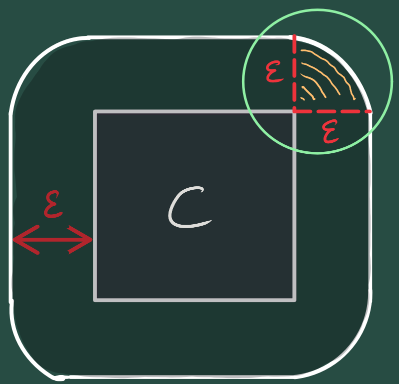


Can we compute Vol_K of a polytope?



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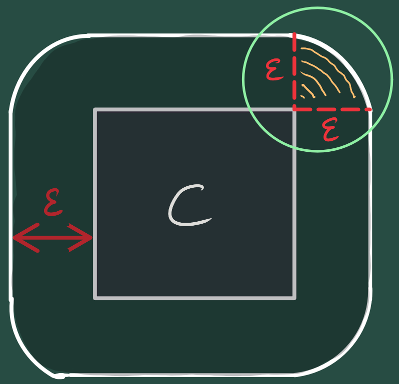

$\Downarrow$

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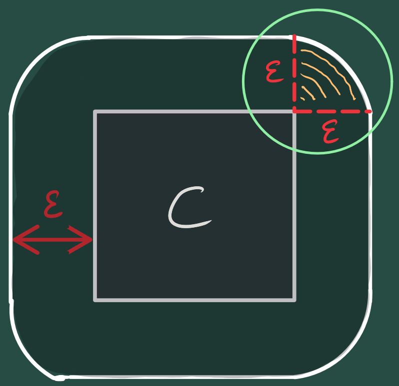

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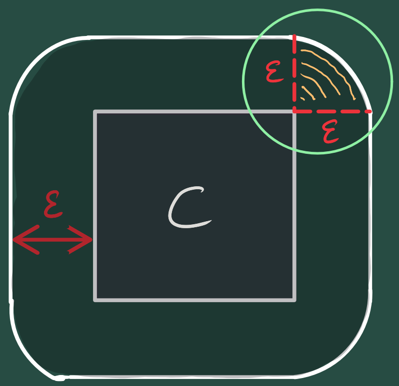

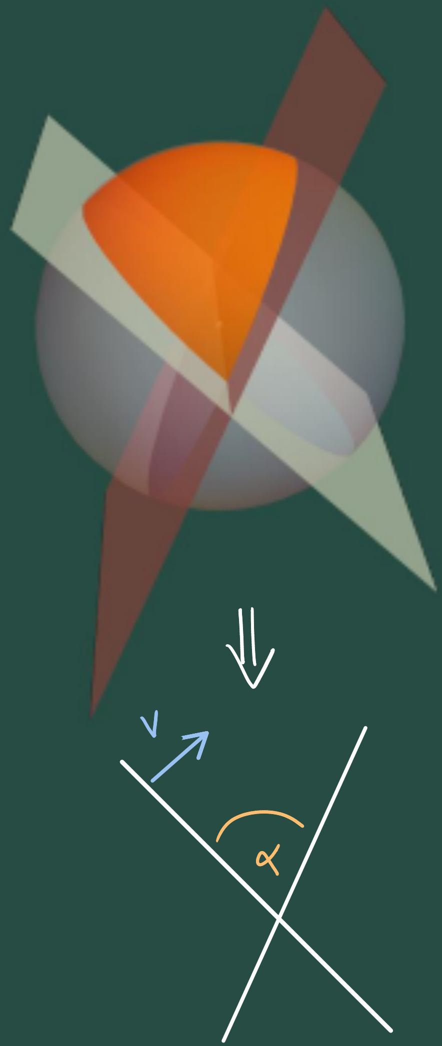

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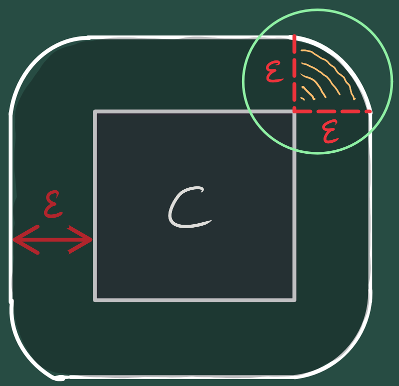

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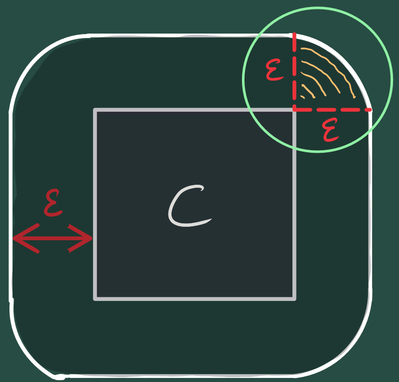

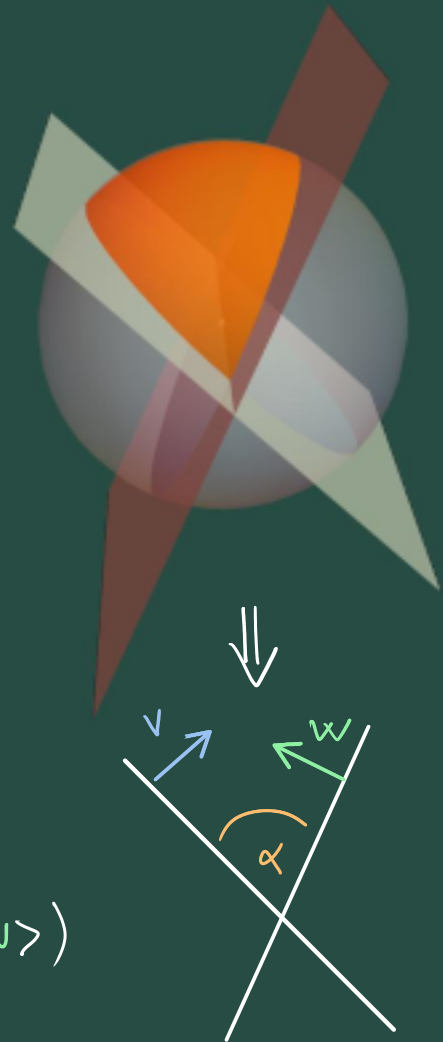

$$\alpha = \arccos(-\langle v, w \rangle)$$

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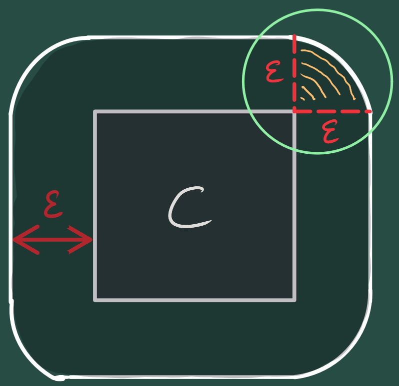

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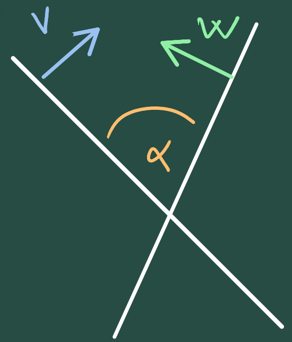

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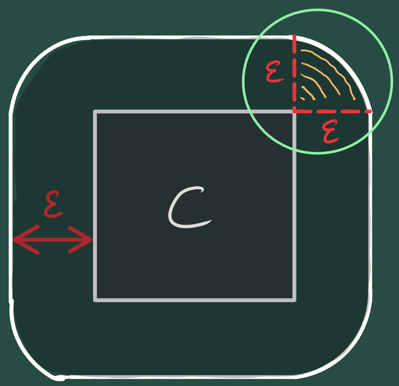

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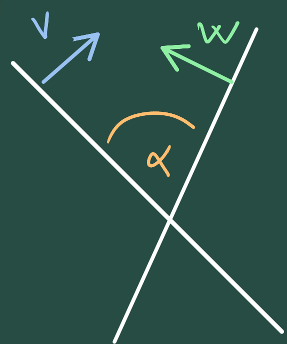

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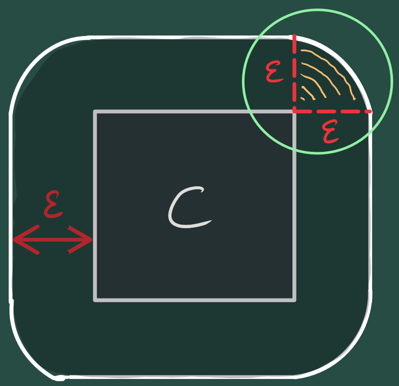

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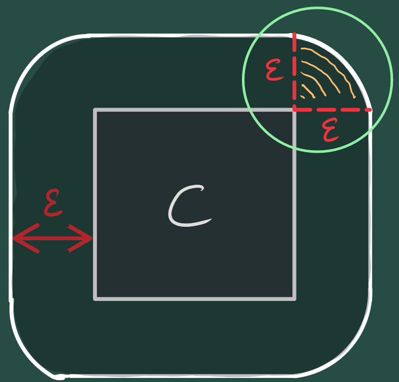

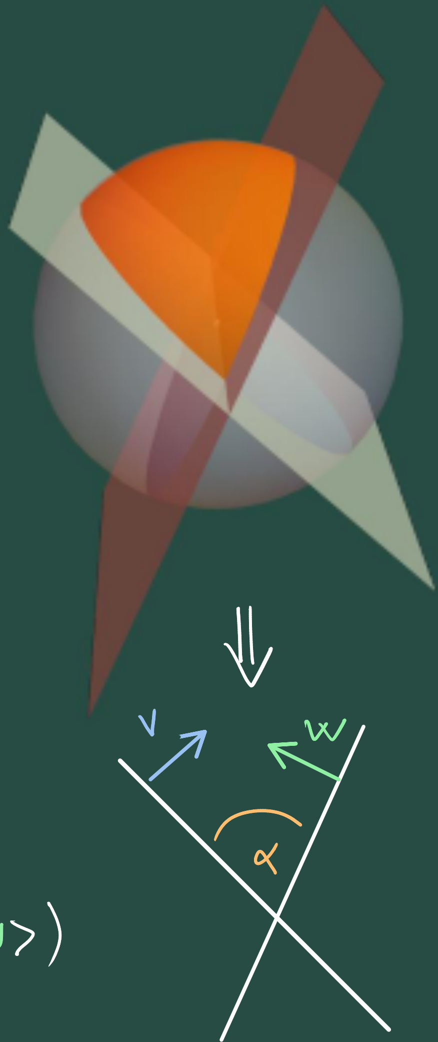

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∃ formula for the volume ratio of the part of the ball

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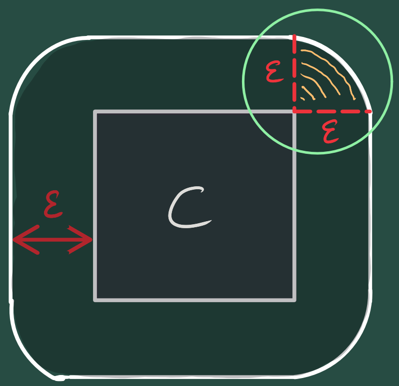

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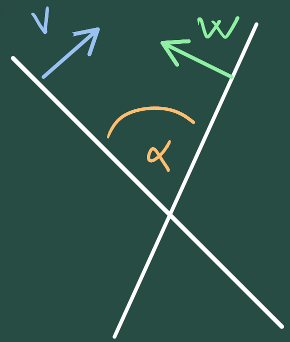

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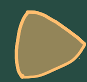
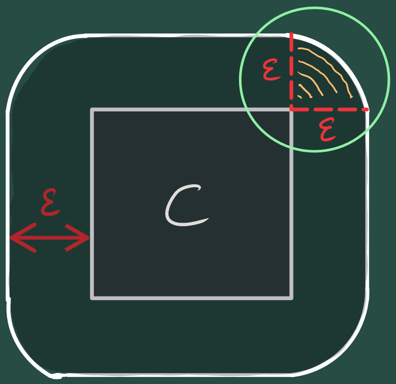
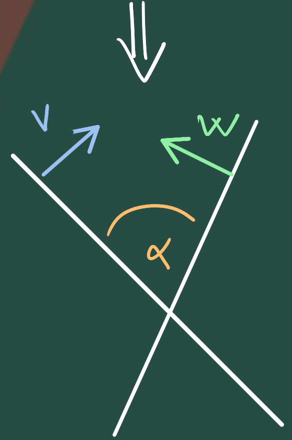
∃ formula for the volume ratio of the  part of the ball

☹ : ∄ known formula for the case of 4 hyperplanes

Can we compute Vol_k of a polytope ?





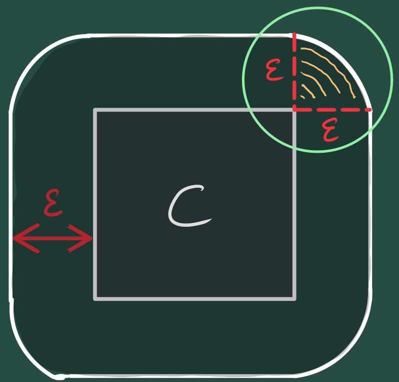
$\alpha = \arccos(-\langle v, w \rangle)$

$\exists$ formula for the volume ratio of the part of the ball

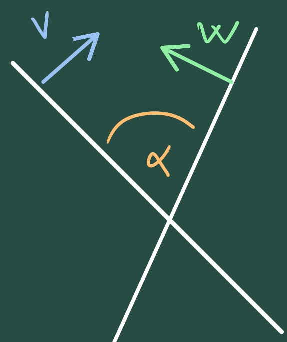
☹ : $\nexists$ known formula for the case of 4 hyperplanes

$\hookrightarrow$ we have alg. for computing Vol_k of a polytope $\subset \mathbb{R}^D$ only for $k = D, D-1, D-2, D-3$

Can we compute Vol_k of a polytope ?



$\Downarrow$



could be worse: perhaps $\nexists$ formula at all

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$\exists$ formula for the volume ratio of the part of the ball

: $\nexists$ known formula for the case of 4 hyperplanes

$\hookrightarrow$ we have alg. for computing Vol_k of a polytope $\subset \mathbb{R}^D$ only for $k = D, D-1, D-2, D-3$

Intrinsic volumes of $\mathcal{P}_d$

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$(D \equiv \dim(\mathcal{P}_d) = d^2 - 1)$

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$$Vol_{D-1}(\mathcal{P}_d) = \frac{\sqrt{d}^{\,d+2}\sqrt{d^2-1}}{(d^2-2)!}$$

(Bengtsson & Ericsson 2004)

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$$Vol_{D-2}(\mathcal{P}_d) = \chi_{2,D}\,\alpha\,\frac{\sqrt{2d^2-d-2}\,(d^2-1)d^{1+d/2}}{4\,(d^2-3)!}$$

$$Vol_{D-3}(\mathcal{P}_d) = \chi_{3,D}\left(\frac{2\sqrt{3d^2-2d-3}\,(d^2-1)(d-2)\sqrt{d}^{\,d+2}}{9(d^2-4)!}\arctan\sqrt{\tan\frac{3\alpha}{4}\tan^3\frac{\alpha}{4}}+\right.$$

$$\left.+\frac{4\sqrt{d^2-d-1}\,(d^2-1)(d-1)\sqrt{d}^{\,d+4}}{3\,(d^2-4)!}\arctan\sqrt{\tan(\frac{\alpha}{2}+\frac{\beta}{4})\tan(\frac{\alpha}{2}-\frac{\beta}{4})\tan^2(\frac{\beta}{4})}\right)$$

where $\alpha = \arccos\left(1-\frac{d}{d^2-1}\right),\;\; \beta = \arccos\left(1-\frac{2d}{d^2-1}\right)$

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☹ : difficult computation resulting long and complicated formulas

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: difficult computation resulting long and complicated formulas

: numerically confirmed with Monte-Carlo simulations to 3 digits (up to d≤5)

Intrinsic volumes of $S_1^+(\mathbb{C}^d)$

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$(D \equiv \dim(\mathcal{S}_1^+(\mathbb{C}^d)) = d^2 - 1)$

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$$Vol_D\big(\mathcal{S}_1^+(\mathbb{C}^d)\big) \;=\; \sqrt{d}\,(2\pi)^{\frac{d(d-1)}{2}}\,\frac{\Gamma(1)\cdots\Gamma(d)}{\Gamma(d^2)}$$

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Their method can be used to compute *two further* intrinsic volumes:

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Their method can be used to compute *two further* intrinsic volumes:

$$Vol_{D-2}\big(\mathcal{S}_1^+(\mathbb{C}^d)\big) = (d-1)d^{\frac{3}{2}}(2\pi)^{\frac{d(d-1)}{2}}\,\frac{\Gamma(1)\cdots\Gamma(d)}{\Gamma(d^2-2)}$$

$$Vol_{D-3}\big(\mathcal{S}_1^+(\mathbb{C}^d)\big) = d(d-1)^{\frac{3}{2}}(2\pi)^{\frac{d(d-1)}{2}}\,\frac{\Gamma(1)\cdots\Gamma(d+1)}{\Gamma(d)\Gamma(d^2-3)}$$

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☹

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☹ : $Vol_{D-j}(\mathcal{P}_d) < Vol_{D-j}\big(\mathcal{S}_1^+(\mathbb{C}^d)\big)$ for every j=0,1,2,3 and d=2,3,4,5,..

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: $Vol_{D-j}(\mathcal{P}_d) \;<\; Vol_{D-j}\big(\mathcal{S}_1^+(\mathbb{C}^d)\big)$ for every j=0,1,2,3 and d=2,3,4,5,..

: the two further volumes can be used to exclude the existence of *some* configuations (although not the one corresponding to a compl. system of mutually unbiased bases)

An example

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Let $k = 0, 1, 2$ or 3, $d \geq 5$.

$D \equiv d^2 - 1$

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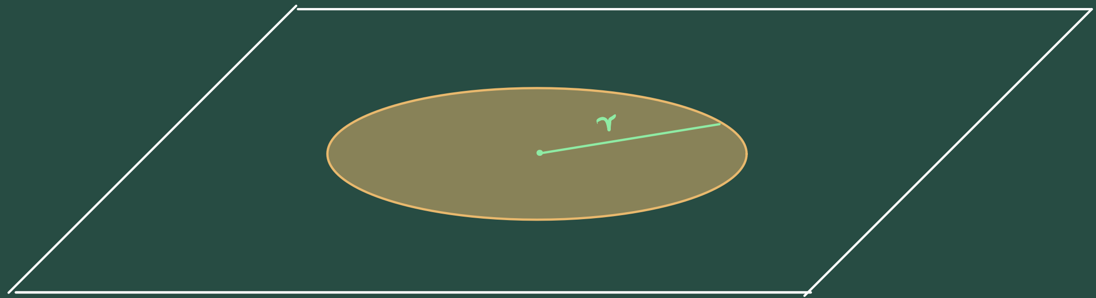


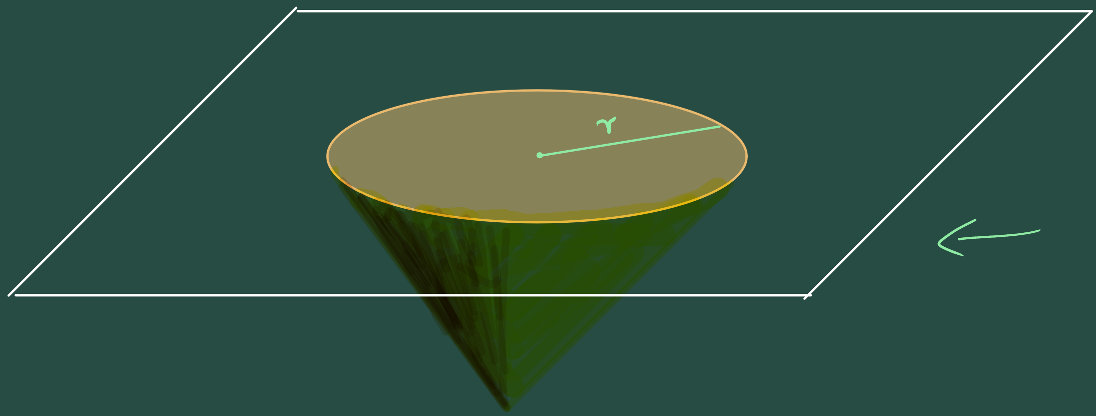
A $D-1-k$ dim. ball
with radius $r = \frac{1}{\sqrt{2}}$

An example

Let $k = 0, 1, 2$ or 3, $d \geq 5$.

$D \equiv d^2 - 1$

A $D-1-k$ dim. ball with radius $r = \frac{1}{\sqrt{2}}$



A $D-k$ dim. cone with height $h = \sqrt{\frac{1}{2} - \frac{1}{d}}$

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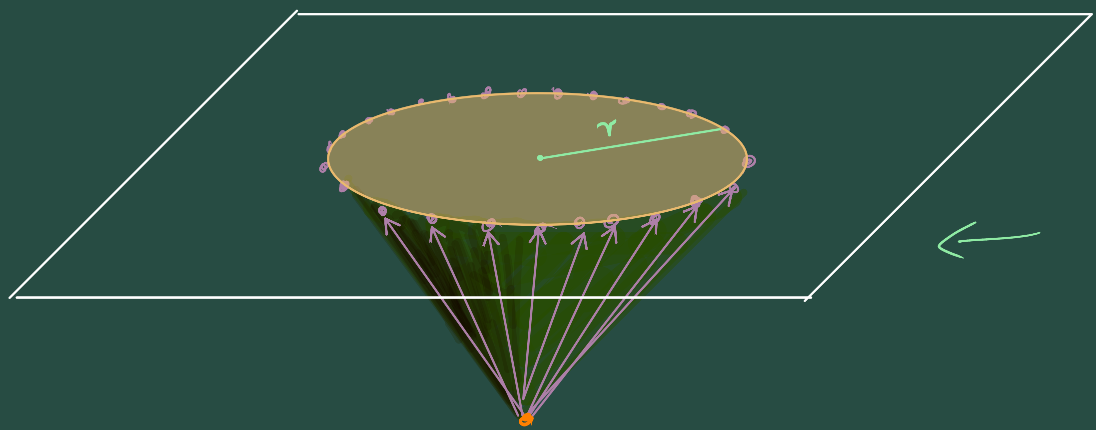

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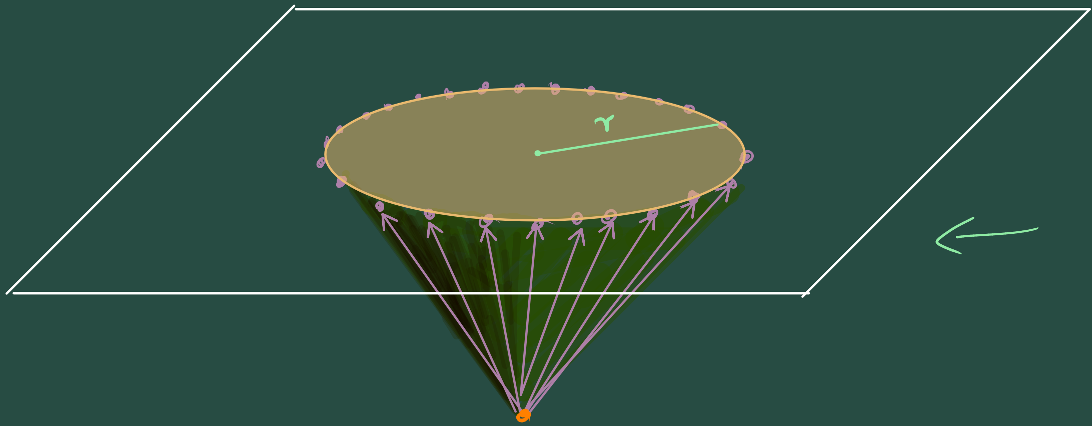

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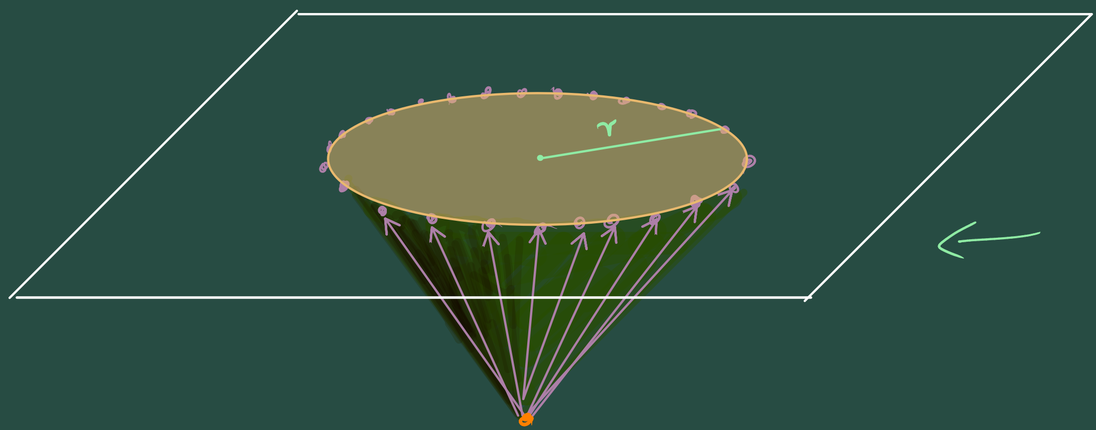

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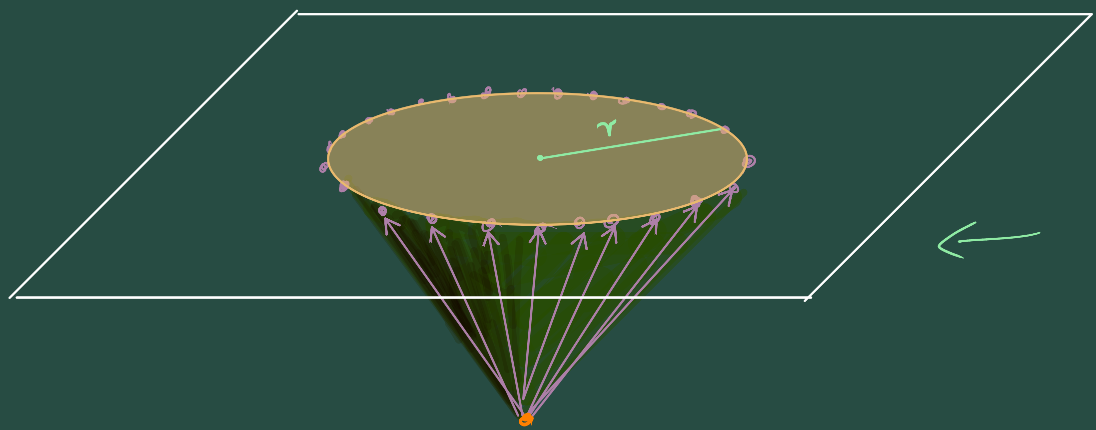

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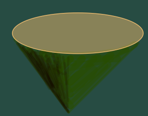
Consider ? $\exists$  $\hookrightarrow S_1^+(\mathbb{C}^d)$.

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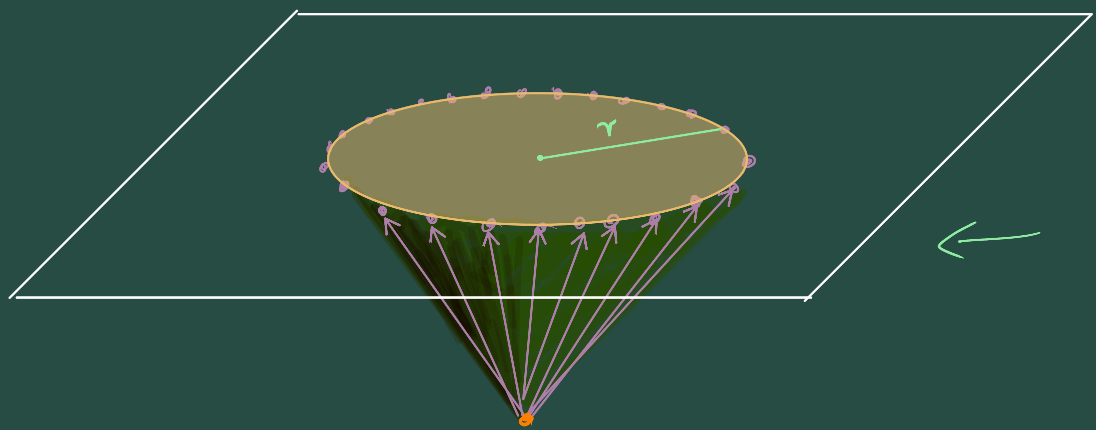

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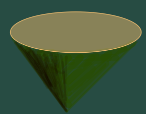
Consider ? $\exists$  $\hookrightarrow S_1^+(\mathbb{C}^d)$.

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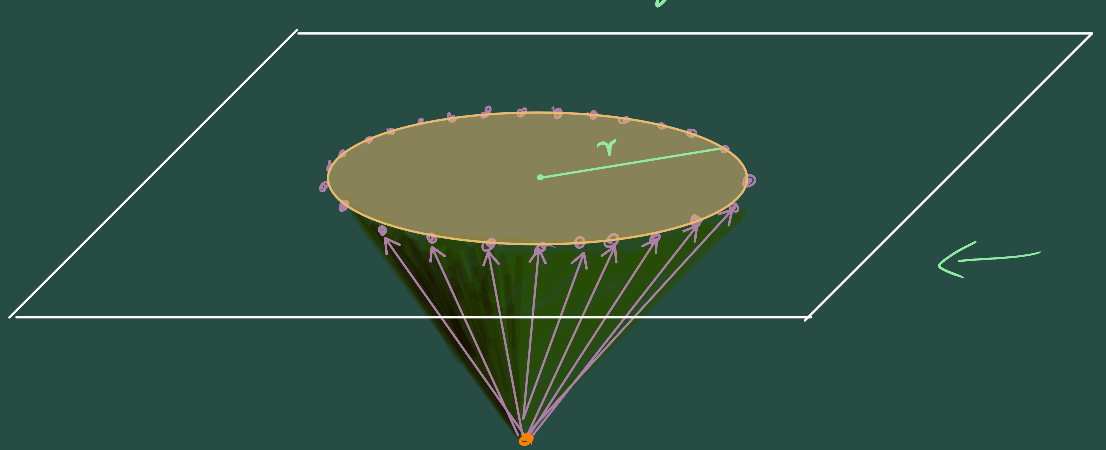

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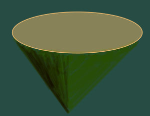
Consider ? $\exists$  $\hookrightarrow S_1^+(\mathbb{C}^d)$.

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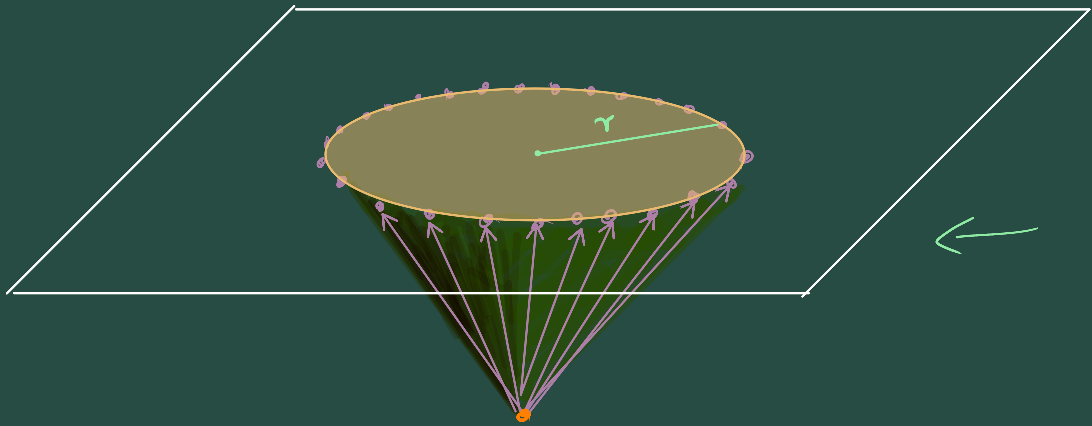

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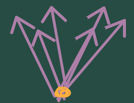
Question: ? $\exists$  $\hookrightarrow S_1^+(\mathbb{C}^d)$

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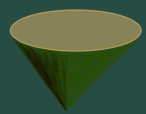
Consider ? $\exists$  $\hookrightarrow S_1^+(\mathbb{C}^d)$.

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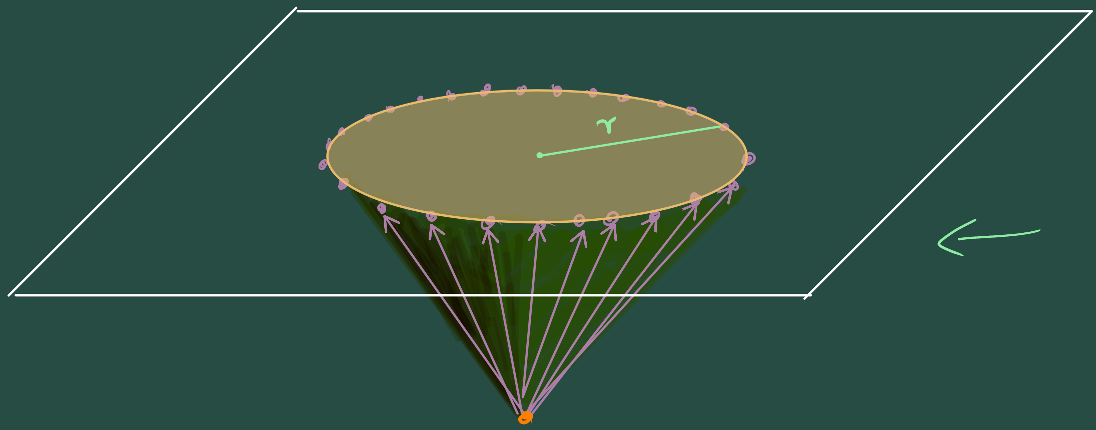

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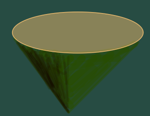
Consider $? \, \exists$  $\hookrightarrow S_1^+(\mathbb{C}^d)$.

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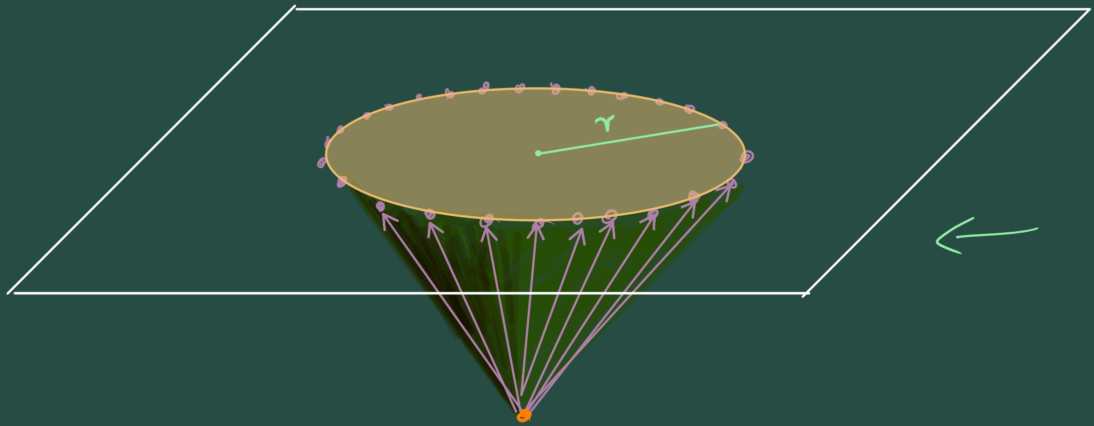

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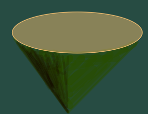
Question : ? $\exists$  $\hookrightarrow S^+_1(\mathbb{C}^d)$ T : ✓

Consider ? $\exists$  $\hookrightarrow S^+_1(\mathbb{C}^d)$.

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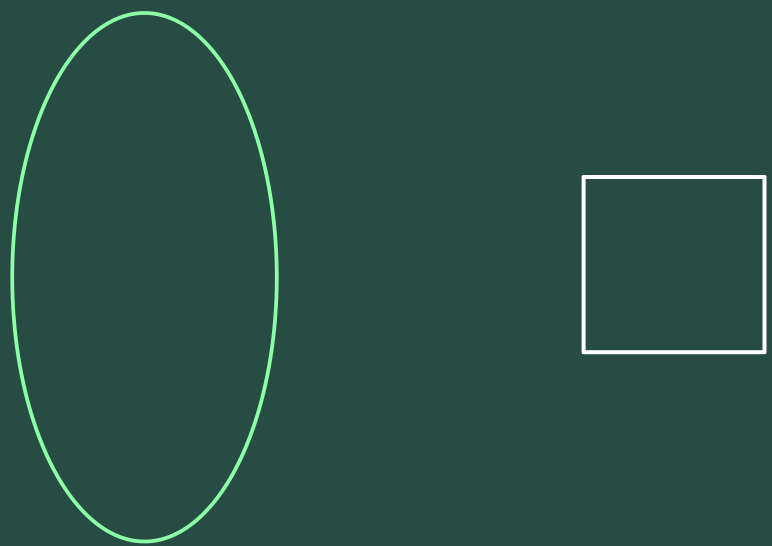

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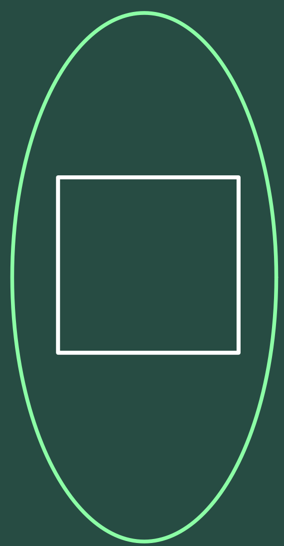

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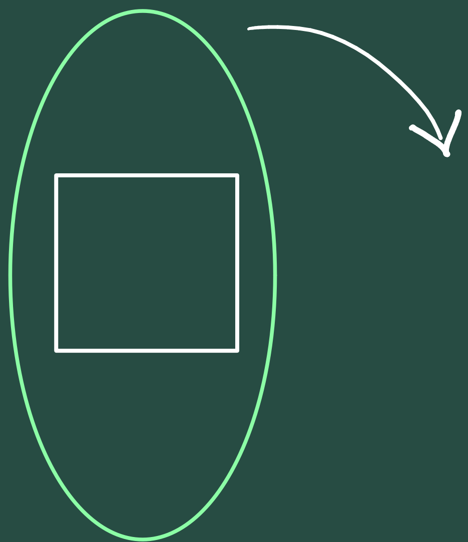

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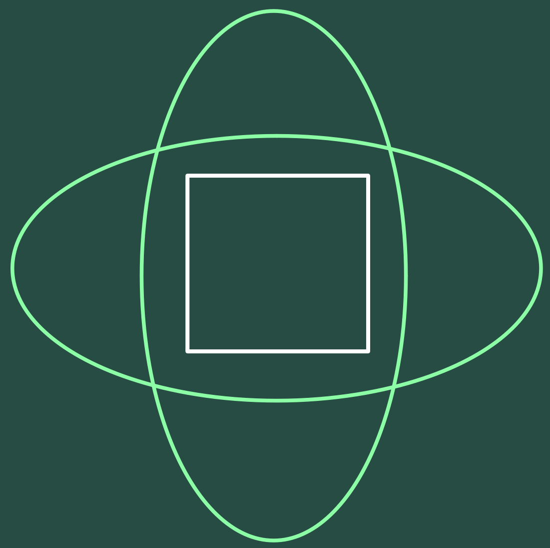

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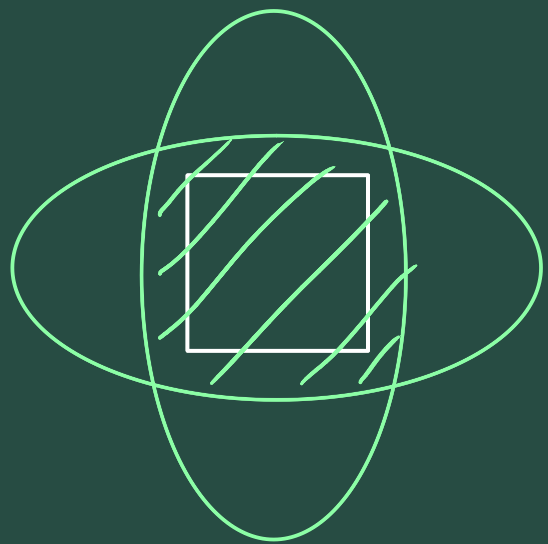

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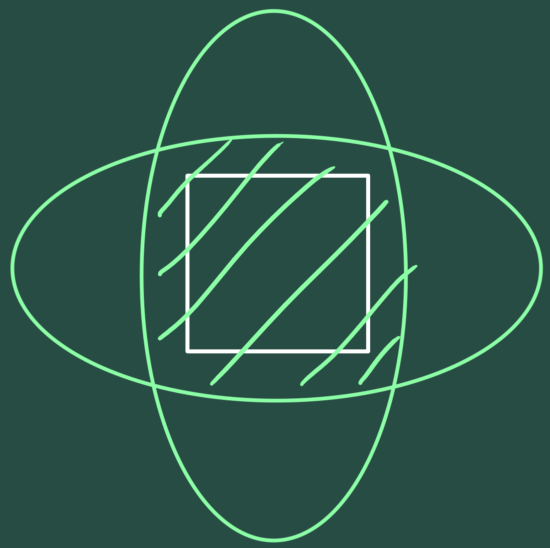

Thank you for your attention!