Constructing directed strongly regular graphs using their orbit matrices and genetic algorithm

Tin Zrinski

Faculty of Mathematics, University of Rijeka, Croatia

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Joint work with Dean Crnković and Andrea Švob

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Introduction and preliminaries

We propose a method of constructing **directed strongly regular graphs (DSRGs)** which combines **genetic algorithm** and a method for constructing DSRGs with prescribed automorphism group using **orbit matrices**.

We apply this method to construct some directed strongly regular graphs on 36, 52, 55 and 60 vertices.

Genetic algorithm

Genetic algorithms (GA) are search and optimization heuristic population based methods which are inspired by the natural evolution process. In each step of the algorithm, a subset of the whole solution space, called **population**, is being treated. The population consists of **individuals**, and each individual has **genes** that can be mutated and altered.

Every individual represents a possible solution (optimum), which is evaluated using the **fitness function**. In each iteration of the algorithm, a certain number of best-ranked individuals - **parents** is selected to create new better individuals - **children**. Children are created by a certain type of recombination - **crossover** and they replace the worst-ranked individuals in the population, providing convergence to the local optimum. After children are obtained, a **mutation** operator is allowed to occur and the next generation of the population is created. The process is iterated until the evolution condition terminates.

Definition of directed strongly regular graphs

Duval[1] introduced the following definition:

Definition

Let $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ be a simple directed k-regular graph with v vertices. We say that Γ is a **directed strongly regular graph** with parameters (v, k, t, λ, μ) if the number of directed paths of length 2 from a vertex x to a vertex y is

- λ if there is an arc from x to y,
- μ if there is no arc from x to y,
- t if $x{=}y$.

Directed strongly regular graphs with parameters (v, k, t, λ, μ) are denoted by $DSRG(v, k, t, \lambda, \mu)$.

[1] A. M. Duval, A directed graph version of strongly regular graphs, J. Combin. Theory Ser. A 47 (1988), 71–100.

Orbit matrices of directed strongly regular graphs

Let Γ be a $DSRG(v,k,t,\lambda,\mu)$ and A be its adjacency matrix. Suppose an automorphism group G of Γ partitions the set of vertices $\mathcal{V}$ into b orbits $O_1,\ldots,O_b$, of sizes $n_1,\ldots,n_b$, respectively. The orbits divide A into submatrices $[A_{ij}]$, where A_{ij} is the adjacency matrix of vertices in O_i versus those in O_j. We define matrices $C=[c_{ij}]$ and $R=[r_{ij}]$, $1\le i,j\le b$, such that c_{ij} is the column sum of A_{ij} and r_{ij} is the row sum of A_{ij}. Clearly, $R=C^T$. The matrix R is the **row orbit matrix** of the graph Γ with respect to G, and the matrix C is the **column orbit matrix** of the graph Γ with respect to G.

Row orbit matrices of directed strongly regular graphs

Theorem (D. Crnković, A. Švob, TZ, 2024)

Let Γ be a DSRG(v, k, t, λ, μ) and let G be an automorphism group of Γ. Further, let $O_1, O_2, \ldots, O_b$ be the G-orbits on vertex set of Γ, and let $|O_i| = n_i$, $i = 1, \ldots, b$, be the corresponding orbit lengths. If $R = [r_{ij}]$ is the **row orbit matrix** of Γ with respect to the action of G, then the following hold

$$0 \leq r_{ij} \leq n_j, \text{ for } 1 \leq i, j \leq b, \tag{1}$$

$$0 \leq r_{ii} \leq n_i - 1, \text{ for } 1 \leq i \leq b, \tag{2}$$

$$\sum_{j=1}^{b} r_{ij} = k, \text{ for } 1 \leq i \leq b, \tag{3}$$

$$\sum_{s=1}^{b} r_{is} r_{sj} = \delta_{ij}(t - \mu) + r_{ij}\lambda + (n_j - r_{ij})\mu, \text{ for } 1 \leq i, j \leq b, \tag{4}$$

where δ_{ij} is the Kronecker delta.

Definition of row orbit matrices of directed strongly regular graphs

Definition

Let n_i, $i = 1, \ldots, b$, be positive integers such that $\sum_{i=1}^{b} n_i = v$. A $(b \times b)$-matrix $R = [r_{ij}]$, where r_{ij}, $1 \leq i, j \leq b$, are non-negative integers satisfying conditions

$$0 \leq r_{ij} \leq n_j, \text{ for } 1 \leq i, j \leq b, \quad 0 \leq r_{ii} \leq n_i - 1, \text{ for } 1 \leq i \leq b,$$

$$\sum_{j=1}^{b} r_{ij} = k, \text{ for } 1 \leq i \leq b, \quad \sum_{i=1}^{b} \frac{n_i}{n_j} r_{ij} = k, \text{ for } 1 \leq j \leq b,$$

$$\sum_{s=1}^{b} r_{is} r_{sj} = \delta_{ij}(t - \mu) + r_{ij}\lambda + (n_j - r_{ij})\mu, \text{ for } 1 \leq i, j \leq b,$$

where δ_{ij} is the Kronecker delta, is a **row orbit matrix** associated with a DSRG(v, k, t, λ, μ) and the distribution of orbit lengths $(n_1, n_2, \ldots n_b)$.

Remark

Remark

Orbit matrices from the definitions may or may not correspond to a directed strongly regular graph with parameters (v, k, t, λ, μ).
Those matrices can be used for constructing directed strongly regular graphs with a presumed automorphism group in a similar way as orbit matrices of 2-designs are used for a construction of 2-designs and orbit matrices of strongly regular graphs are used for a construction of strongly regular graphs.
In this work, we use orbit matrices of DSRGs, together with a genetic algorithm, to construct DSRGs which have a presumed group of automorphisms.

Combining OM and GA to construct DSRGs

Let (v, k, t, λ, μ) be admissible parameters for a directed strongly regular graph. Further, let G be a finite group, and let $(n_1, n_2, \ldots, n_b)$ be a possible vertex orbit lengths distribution under the action of the group G on a DSRG(v, k, t, λ, μ).

The distribution $(n_1, n_2, \ldots, n_b)$ may arise from the actual action of G on a directed strongly regular graph with parameters (v, k, t, λ, μ), or it can be a potential orbit lengths distribution for an action of G.

Furthermore, let M be a row orbit matrix for a DSRG(v, k, t, λ, μ) and the distribution of orbit lengths $(n_1, n_2, \ldots, n_b)$.

Combining OM and GA to construct DSRGs

Our goal is to construct adjacency matrices of directed strongly regular graphs with parameters (v, k, t, λ, μ) which admit the action of an automorphism group isomorphic to G with the distribution of orbit lengths $(n_1, n_2, \ldots, n_b)$, such that this action produces the orbit matrix M.

In this work, we construct directed strongly regular graphs by prescribing actions of cyclic groups of orders 2, 3, 5, 11 or 13. While indexing the orbit matrix, we employ a genetic algorithm, which means that our search will not be exhaustive. In other words, this method may not produce all directed strongly regular graphs, up to isomorphism, which can be constructed from the given orbit matrix M.

While testing the algorithm on examples of known directed strongly regular graphs for various parameters (v, k, t, λ, μ), we have determined optimal values of parameters of a genetic algorithm for a construction of DSRGs.

Combining OM and GA to construct DSRGs

In this construction, the **individuals** are adjacency matrices of simple directed k-regular graphs on v vertices which allow an action of group G corresponding to the row orbit matrix M.
In other words, the individuals are actually $(0,1)$-matrices of dimension $v \times v$ with zeroes on the diagonal, whose row sums and column sums are k and which allow the action of G with the orbit lengths distribution $(n_1, n_2, \ldots, n_b)$, and the orbit matrix M.
The goal is to start with some starting population that consists of a specified number of such randomly generated individuals and, by using a genetic algorithm, to construct an individual that is the adjacency matrix of a DSRG(v, k, t, λ, μ).

Combining OM and GA to construct DSRGs

A **gene** of such an individual is the union of rows of the adjacency matrix which correspond to one vertex orbit.
A **bit** is a submatrix which is obtained as the intersection of rows and columns of the adjacency matrix which correspond to two vertex orbits. That means that bits are submatrices of the adjacency matrix that correspond to an element of the orbit matrix. Every bit is determined by its first row, whereas the other rows are uniquely determined by the group G acting on the first row.
For some elements of the orbit matrix, e.g. for the element of a row or column that correspond to a vertex fixed by G, the corresponding bits are uniquely determined. Such bits are called **fixed bits**. The bits that are not fixed, i.e. those bits that coincide with the elements of the orbit matrix where there exist more possibilities to construct the corresponding bit, are called non-fixed bits. Similarly, we have fixed and non-fixed genes.

Combining OM and GA to construct DSRGs

The **fitness function** is defined as follows. For every two distinct vertices v_i and v_j, we denote by x_{ij} the number of directed paths of length 2 from v_i to v_j i.e., the dot product of the i^{th} row and j^{th} column of the adjacency matrix.

The fitness function is

$$\sum_{v_i, v_j \in \mathcal{V}} \begin{cases} min\{x_{ij}, t\}, & \text{if } v_i = v_j, \\ min\{x_{ij}, \lambda\}, & \text{if there is an arc from } v_i \text{ to } v_j, \\ min\{x_{ij}, \mu\}, & \text{if there is no arc from } v_i \text{ to } v_j, \end{cases}$$

where $\mathcal{V}$ is the set of vertices of a graph.

Combining OM and GA to construct DSRGs

For such a fitness function, an individual will be an adjacency matrix of a DSRG(v, k, t, λ, μ) if and only if its fitness is equal to

$$vt + vk\lambda + v(v - k - 1)\mu.$$

This is because in a DSRG(v, k, t, λ, μ) each of v vertices has t directed paths of length 2 to itself; for every vertex x there are k arcs $x \to y$, and for each of these arcs there are λ directed paths of length 2 from x to y; finally, each vertex x has no arcs from itself to $(v - k - 1)$ vertices and there are μ directed paths of length 2 from x to each of these $(v - k - 1)$ vertices.
With such a fitness function, the problem of finding an optimal solution, i.e., an adjacency matrix of a directed strongly regular graph, is a **maximization problem**.

Combining OM and GA to construct DSRGs

The **crossover** is performed such that we replace the genes of the first parent at some positions with the genes of the second parent at the same positions, and vice versa.

The **mutation** is defined such that we replace some bits of an individual with new, randomly generated bits. That means that we randomly permute the first row in a bit and by the action of G, its other rows are determined.

Sometimes it can happen that a population is in a stagnation, which means that it becomes trapped in a local optimum. If this happens, the whole process is **restarted**. In the algorithm, there are two types of resets, partial and complete. A partial reset is performed in such a way that a certain (predetermined) percentage of the individuals that are ranked as the best, stay in the population, and new randomly generated individuals are added to the population. If the algorithm does not produce a solution after a certain (predetermined) number of partial resets, then the only thing that can be done is the complete reset, i.e. the algorithm starts with a completely new population.

$DSRG(36, 10, 5, 2, 3)$

From one $DSRG(36, 10, 5, 2, 3)$ by L. K. Jørgensen[2] we obtain:

$\lvert \mathrm{Aut}(\Gamma) \rvert$	$\mathrm{Aut}(\Gamma)$	# non-isom.
72	$S_3 \wr 2$	2
36	$S_3 \times S_3$	6
18	$3 : S_3$	6
18	$3 \times S_3$	16
12	D_6	58
9	3^2	16
8	D_4	7
6	S_3	71
6	6	158
4	4	16
4	2^2	119
3	3	193
2	2	235

Table: 903 DSRG(36, 10, 5, 2, 3)

[2] L. K. Jørgensen, New mixed Moore graphs and directed strongly regular graphs, Discrete Math. 338 (2015) 1011-1016.

$DSRG(52,12,3,2,3)$ and $DSRG(52,15,6,5,4)$

From two $DSRG(52,12,3,2,3)$ constructed by A. Švob we obtain:

$\|Aut(\Gamma)\|$	$Aut(\Gamma)$	# non-isom.
5616	$PSL(3,3)$	2
13	13	3
4	4	1
4	2^2	16
3	3	79
2	2	466

Table: 567 DSRG$(52,12,3,2,3)$

From one $DSRG(52,15,6,5,4)$ constructed by A. Švob we obtain:

$\|Aut(\Gamma)\|$	$Aut(\Gamma)$	# non-isom.
5616	$PSL(3,3)$	1
4	2^2	1
3	3	243
2	2	25

Table: 270 DSRG$(52,15,6,5,4)$

$DSRG(55, 20, 8, 6, 8)$ and $DSRG(55, 24, 12, 11, 10)$

From two $DSRG(55, 20, 8, 6, 8)$ constructed by A. Švob we obtain:

$\lvert Aut(\Gamma)\rvert$	$Aut(\Gamma)$	# non-isom.
660	$PSL(2, 11)$	2
8	D_4	9
3	3	3
2	2	97

Table: 111 DSRG$(55, 20, 8, 6, 8)$

From one $DSRG(55, 24, 12, 11, 10)$ constructed by A. Švob we obtain:

$\lvert Aut(\Gamma)\rvert$	$Aut(\Gamma)$	# non-isom.
660	$PSL(2, 11)$	1
3	3	23
2	2	3

Table: 27 DSRG$(55, 24, 12, 11, 10)$

DSRG on 60 vertices

- Orbit matrices of two DSRG$(60, 18, 6, 4, 6)$ obtained[3] from the group S_6, lead us to 54 DSRGs for presumed action of cyclic groups of orders 2, 3 and 5.
- From two DSRG$(60, 20, 8, 4, 8)$ from the group S_6 we obtain the 73 DSRGs for presumed action of cyclic groups of orders 2, 3 and 5.
- From two DSRG$(60, 23, 11, 10, 8)$ from the group S_6 we obtain 24 DSRGs for presumed action of groups cyclic groups of orders 2 and 3.
- From two DSRG$(60, 29, 17, 16, 12)$ from the group S_6 we obtain 112 DSRGs for presumed action of cyclic groups of orders 2, 3 and 5.
- From two DSRG$(60, 30, 18, 12, 18)$ from the group S_6 we obtain 77 DSRGs for presumed action of cyclic groups of orders 2, 3 and 5.

[3] D. Crnković, V. Mikulić Crnković, A. Švob, On some transitive combinatorial structures constructed from the unitary group U(3,3), J. Statist. Plann. Inference 144 (2014), 19–40.

Thank you for your attention!