CENSUS OF SYMMETRIC AND SKEW HADAMARD MATRICES OF ORDER $4v$ FOR ODD $v < 2500$

DRAGOMIR Ž. ĐOKOVIĆ
UNIVERSITY OF WATERLOO,
DEPARTMENT OF PURE MATHEMATICS,
WATERLOO, ONTARIO, N2L 3G1, CANADA

AND JENNIFER SEBERRY
UNIVERSITY OF WOLLONGONG,
SCHOOL OF COMPUTING AND INFORMATION TECHNOLOGY,
FACULTY OF ENGINEERING AND INFORMATION SCIENCE,
WOLLONGONG, NSW 2522, AUSTRALIA

Abstract. An up-to-date listing of the existence of symmetric and skew Hadamard matrices of order $4v$ for odd $v < 2500$ is given.

1. Introduction

As the literature on Hadamard matrices is huge and overwhelming, it is very hard to know for which orders such matrices exist. For instance, for about 34 years in several books and many papers it was asserted that it is known how to construct skew Hadamard matrix of order 292. Once noticed a construction was quickly found (in a day or two).

In this article our objective is to provide reliable information about existence of symmetric and skew Hadamard matrices of order 4v with v odd and less than 2500. We first list such orders from well known infinite series. After that we list the sporadic cases, constructed mostly ad hoc using computer searches. For each of these cases we give a reference where the construction can be found. In spite of our best-efforts,it is likely we missed some orders. The restriction that v is odd is an important simplication which made our job much easier: in this case it I essential because it is well known that the size of such matrices Hadamard, symmetric Hadamard and skew Hadamard) can be doubled.

2. Symmetric Hadamard matrices of order $4v$ with odd $v < 2500$

For each of the series of symmetric Hadamard matrices listed below we give the number of their orders $4v$ with $v < 2500$. In the column of "cumulative totals" we give the number of these orders (avoiding repetition) in the chosen series and all previous series. For each series we list only the values of v which do not occur in the previous ones.

Paley-1 series: (see [25]): 356 matrices, $v = (q+1)/2$, $q \equiv 1 \pmod 4$ a prime power.

Series	Cumulative Total
1, 3, 5, 7, 9, 13, 15, 19, 21, 25, 27, 31, 37, 41, 45, 49, 51, 55, 57, 61, 63, 69, 75, 79, 85, 87, 91, 97, 99, 115, 117, 121, 129, 135, 139, 141, 145, 147, 157, 159, 169, 175, 177, 181, 187, 195, 199, 201, 205, 211, 217, 225, 229, 231, 255, 261, 265, 271, 279, 285, 289, 297, 301, 307, 309, 313, 321, 327, 331, 337, 339, 351, 355, 365, 367, 379, 381, 385, 387, 399, 405, 411, 415, 421, 427, 429, 439, 441, 465, 469, 471, 477, 481, 489, 499, 505, 507, 511, 517, 525, 531, 535, 547, 549, 555, 559, 565, 577, 591, 597, 601, 607, 609, 615, 619, 625, 639, 645, 649, 651, 661, 681, 685, 687, 691, 705, 715, 717, 727, 741, 745, 747, 775, 777, 799, 801, 805, 807, 811, 819, 829, 835, 841, 847, 849, 855, 861, 867, 871, 877, 889, 895, 901, 925, 931, 937, 939, 945, 951, 957, 967, 975, 987, 997, 999, 1009, 1015, 1027, 1035, 1041, 1045, 1057, 1065, 1069, 1071, 1077, 1081, 1099, 1105, 1107, 1111, 1119, 1135, 1137, 1141, 1147, 1149, 1155, 1167, 1171, 1179, 1189, 1191, 1195, 1197, 1201, 1209, 1219, 1221, 1237, 1239, 1261, 1275, 1279, 1297, 1305, 1309, 1311, 1317, 1329, 1339, 1345, 1347, 1357, 1365, 1371, 1375, 1377, 1389, 1395, 1399, 1401, 1405, 1417, 1419, 1429, 1431, 1449, 1455, 1459, 1477, 1479, 1485, 1501, 1519, 1521, 1525, 1531, 1545, 1555, 1561, 1563, 1569, 1585, 1591, 1605, 1609, 1611, 1615, 1627, 1629, 1651, 1657, 1665, 1681, 1687, 1695, 1707, 1717, 1725, 1729, 1731, 1735, 1741, 1759, 1765, 1767, 1771, 1779, 1791, 1797, 1807, 1809, 1819, 1837, 1839, 1849, 1851, 1855, 1861, 1867, 1881, 1885, 1897, 1899, 1911, 1917, 1927, 1939, 1941, 1945, 1959, 1965, 1995, 2001, 2007, 2011, 2025, 2029, 2037, 2047, 2065, 2067, 2077, 2079, 2089, 2101, 2109, 2115, 2121, 2127, 2131, 2137, 2145, 2149, 2169, 2175, 2179, 2187, 2199, 2205, 2211, 2221, 2229, 2241, 2245, 2247, 2257, 2259, 2275, 2281, 2299, 2311, 2319, 2325, 2329, 2337, 2361, 2365, 2367, 2395, 2397, 2401, 2407, 2409, 2431, 2439, 2445, 2455, 2457, 2467, 2469, 2479, 2485, 2487, 2497	356

Note: The Turyn series of Williamson matrices [36] gives also the symmetric Hadamard matrices (by using the propus array [28] instead of the Williamson array) of the same orders as the Paley-1 series.

XXSW series (see [40, 6]): 316 matrices, $4v = q + 1$, $q \equiv 3 \pmod 8$ a prime power.

We list below only the values v in this series which do not occur in the previous series.

Series	Cumulative Total
$11, 17, 33, 35, 53, 71, 77, 83, 95, 105, 111, 123, 125, 131, 137, 143, 155, 161, 165, 171, 173, 185, 197, 203, 207, 215, 221, 227, 237, 243, 263, 273, 281, 291, 293, 315, 323, 333, 357, 363, 371, 375, 383, 393, 395, 407, 417, 425, 431, 437, 447, 453, 467, 483, 495, 497, 501, 503, 521, 533, 545, 551, 561, 563, 567, 585, 587, 593, 603, 617, 633, 635, 665, 671, 675, 677, 683, 701, 711, 713, 735, 743, 753, 755, 767, 771, 791, 797, 813, 815, 825, 827, 831, 833, 837, 843, 873, 875, 885, 887, 893, 911, 915, 923, 935, 963, 977, 981, 983, 1001, 1005, 1007, 1013, 1023, 1025, 1053, 1055, 1061, 1085, 1091, 1113, 1121, 1127, 1131, 1151, 1161, 1163, 1173, 1181, 1233, 1247, 1251, 1253, 1263, 1265, 1277, 1287, 1293, 1295, 1307, 1331, 1337, 1355, 1361, 1383, 1391, 1413, 1415, 1421, 1445, 1457, 1461, 1463, 1467, 1481, 1497, 1503, 1511, 1517, 1523, 1533, 1541, 1551, 1553, 1575, 1581, 1595, 1607, 1613, 1623, 1637, 1641, 1643, 1655, 1673, 1691, 1701, 1715, 1721, 1727, 1737, 1743, 1755, 1757, 1761, 1803, 1805, 1811, 1821, 1827, 1833, 1853, 1863, 1865, 1875, 1877, 1887, 1901, 1923, 1925, 1931, 1967, 1971, 1977, 1991, 2003, 2015, 2031, 2043, 2045, 2055, 2061, 2073, 2091, 2097, 2105, 2111, 2117, 2135, 2141, 2157, 2177, 2183, 2195, 2201, 2217, 2231, 2243, 2253, 2261, 2265, 2267, 2273, 2297, 2301, 2307, 2321, 2331, 2343, 2351, 2355, 2373, 2385, 2387, 2405, 2411, 2435, 2447, 2451, 2453, 2463, 2465, 2471, 2477, 2481, 2483$	612

Seberry-Balonin series (see [28, Corollary 1] and [2, Introduction]): 56 matrices, $v = q + 2$, $q \equiv 1 \pmod 4$, and $(q + 1)/2$ a prime power or the order of the core of a symmetric conference matrix.

Series	Cumulative Total
$39, 423, 459, 543, 579, 627, 663, 759, 879, 1095, 1203, 1215, 1323, 1659, 1683, 1935, 2019, 2139, 2403, 2475$	632

Turyn C-series (see [4, 36]): 19 matrices, $2v = 1 + (h - 1)^k$, k even, h order of a skew-Hadamard matrix.

Series	Cumulative Total
$113, 613, 761, 1301, 1513, 1985.$	638

Turyn series (see [37]): 4 matrices, $v = 9^i$, $i \geq 0$.

Series	Cumulative Total
$81, 729$	640

Mathon series (see [22]): 3 matrices, $2v = 1 + (q+2)q^2$, $q \equiv 3 \pmod 8$ a prime power and $q+2$ a prime power.

Series	Cumulative Total
$23, 787$	642

Seberry-Whiteman series (see [29]): 4 matrices, $2v = 1 + 5 \cdot 9^k$, k odd.

Series	Cumulative Total
1823	643

Muzichuk-Xiang series (see [24]): 4 matrices, $v = m^4$, $m > 0$ odd integer. New matrices are not found.

There are 12 *sporadic values* of v (those which do not occur in any of the above series) for which a symmetric Hadamard matrix of order $4v$ has been constructed by using computers.

Series	Cumulative Total
$29, 43, 47, 59, 67, 73, 93, 103, 109, 127, 151, 191$	655

For $v = 29, 43$ see [6], for $v = 47, 73$ see [3], for $v = 59$ see [1], for $v = 67, 103, 109, 151$ see [2], for $v = 93$ see [15]. for $v = 127, 191$ see [14].

We list below the 595 odd values of $v < 2500$ for which the existence of symmetric Hadamard matrices of order $4v$ is undecided:

65, 89, 101, 107, 119, 133, 149, 153, 163, 167, 179, 183, 189, 193, 209, 213, 219, 223, 233, 235, 239, 241, 245, 247, 249, 251, 253, 257, 259, 267, 269, 275, 277, 283, 287, 295, 299, 303, 305, 311, 317, 319, 325, 329, 335, 341, 343, 345, 347, 349, 353, 359, 361, 369, 373, 377, 389, 391, 397, 401, 403, 409, 413, 419, 433, 435, 443, 445, 449, 451, 455, 457, 461, 463, 473, 475, 479, 485, 487, 491, 493, 509, 513, 515, 519, 523, 527, 529, 537, 539, 541, 553, 557, 569, 571, 573, 575, 581, 583, 589, 595, 599, 605, 611, 621, 623, 629, 631, 637, 641, 643, 647, 653, 655, 657, 659, 667, 669, 673, 679, 689, 693, 695, 697, 699, 703, 707, 709, 719, 721, 723, 725, 731, 733, 737, 739, 749, 751, 757, 763, 765, 769, 773, 779, 781, 783, 785, 789, 793, 795, 803, 809, 817, 821, 823, 839, 845, 851, 853, 857, 859, 863, 865, 869, 881, 883, 891, 897, 899, 903, 905, 907, 909, 913, 917, 919, 921, 927, 929, 933, 941, 943, 947, 949, 953, 955, 959, 961, 965, 969, 971, 973, 979, 985, 989, 991, 993, 995, 1003, 1011, 1017, 1019, 1021, 1029, 1031, 1033, 1037, 1039, 1043, 1047, 1049, 1051, 1059, 1063, 1067, 1073, 1075, 1079, 1083, 1087, 1089, 1093, 1097, 1101, 1103, 1109, 1115, 1117, 1123, 1125, 1129, 1133, 1139, 1143, 1145, 1153, 1157, 1159, 1165, 1169, 1175, 1177, 1183, 1185, 1187, 1193, 1199, 1205, 1207, 1211, 1213, 1217, 1223, 1225, 1227, 1229, 1231, 1235, 1241, 1243, 1245, 1249, 1255, 1257, 1259, 1267, 1269, 1271, 1273, 1281, 1283, 1285, 1289, 1291, 1299, 1303, 1313, 1315, 1319, 1321, 1325, 1327, 1333, 1335, 1341, 1343, 1349, 1351, 1353, 1359, 1363, 1367, 1369, 1373, 1379, 1381, 1385, 1387, 1393, 1397, 1403, 1407, 1409, 1411, 1423, 1425, 1427, 1433, 1435, 1437, 1439, 1441, 1443, 1447, 1451, 1453, 1465, 1469, 1471, 1473, 1475, 1483, 1487, 1489, 1491, 1493, 1495, 1499, 1505, 1507, 1509, 1515, 1527, 1529, 1535, 1537, 1539, 1543, 1547, 1549, 1557, 1559, 1565, 1567, 1571, 1573, 1577, 1579, 1583, 1587, 1589, 1593, 1597, 1599, 1601, 1603, 1617, 1619, 1621, 1625, 1631, 1633, 1635, 1639, 1645, 1647, 1649, 1653, 1661, 1663, 1667, 1669, 1671, 1675, 1677, 1679, 1685, 1689, 1693, 1697, 1699, 1703, 1705, 1709, 1711, 1713, 1719, 1723, 1733, 1739, 1745, 1747, 1749, 1751, 1753, 1763, 1769, 1773, 1775, 1777, 1781, 1783, 1785, 1787, 1789, 1793, 1795, 1799, 1801, 1813, 1815, 1817, 1825, 1829, 1831, 1835, 1841, 1843, 1845, 1847, 1857, 1859, 1869, 1871, 1873, 1879, 1883, 1889, 1891, 1893, 1895, 1903, 1905, 1907, 1909, 1913, 1915, 1919, 1921, 1929, 1933, 1937, 1943, 1947, 1949, 1951, 1953, 1955, 1957, 1961, 1963, 1969, 1973, 1975, 1979, 1981, 1983, 1987, 1989, 1993, 1997, 1999, 2005, 2009, 2013, 2017, 2021, 2023, 2027, 2033, 2035, 2039, 2041, 2049, 2051, 2053, 2057, 2059, 2063, 2069, 2071, 2075, 2081, 2083, 2085, 2087, 2093, 2095, 2099, 2103, 2107, 2113, 2119, 2123, 2125, 2129, 2133, 2143, 2147, 2151, 2153, 2155, 2159, 2161, 2163, 2165, 2167, 2171, 2173, 2181, 2185, 2189, 2191, 2193, 2197, 2203, 2207, 2209, 2213, 2215, 2219, 2223, 2225, 2227, 2233, 2235, 2237, 2239, 2249, 2251, 2255, 2263, 2269, 2271, 2277, 2279, 2283, 2285, 2287, 2289, 2291, 2293, 2295, 2303, 2305, 2309, 2313, 2315, 2317, 2323, 2327, 2333, 2335, 2339, 2341, 2345, 2347, 2349, 2353, 2357, 2359, 2363, 2369, 2371, 2375, 2377, 2379, 2381, 2383, 2389, 2391, 2393, 2399, 2413, 2415, 2417, 2419, 2421, 2423, 2425, 2427, 2429, 2433, 2437, 2441, 2443, 2449, 2459, 2461, 2473, 2489, 2491, 2493, 2495, 2499

Remark: In his paper [23, Theorem 5] Miyamoto claimed (in the last sentence of Theorem 5) that if $q \equiv 1 \pmod 4$ is an integer and there is a conference matrix of order $q+1$ and a symmetric Hadamard matrix of order $q-1$, then there exists a Hadamard matrix H of order $4q$ and of symmetric block form. This means that when H is partitioned into four blocks of the same size then all four blocks are symmetric matrices, the two diagonal blocks are equal and the sum of the two off-diagonal blocks is 0. However this claim is ignored in the proof of Theorem 5. Further, the two diagonal blocks of the Hadamard matrix H, displayed on p. 93 of his paper, which is used in the proof of the main assertion of his Theorem 5, are definitely not symmetric.

3. Skew Hadamard matrices of order $4v$ with odd $v < 2500$

Paley-3 series: (see [25]): 316 matrices, $v = (q+1)/4$, $q \equiv 3 \pmod 4$ a prime power.

Series	Cumulative Total
1, 3, 5, 7, 11, 15, 17, 21, 27, 33, 35, 41, 45, 53, 57, 61, 63, 71, 77, 83, 87, 95, 105, 111, 117, 123, 125, 131, 137, 141, 143, 147, 155, 161, 165, 171, 173, 185, 197, 203, 207, 215, 221, 227, 237, 243, 255, 263, 273, 281, 291, 293, 297, 315, 321, 323, 327, 333, 357, 363, 365, 371, 375, 381, 383, 393, 395, 405, 407, 417, 425, 431, 437, 447, 453, 467, 477, 483, 495, 497, 501, 503, 507, 521, 525, 533, 545, 547, 551, 561, 563, 567, 585, 587, 593, 603, 615, 617, 633, 635, 645, 665, 671, 675, 677, 683, 701, 705, 711, 713, 735, 741, 743, 753, 755, 767, 771, 791, 797, 801, 813, 815, 825, 827, 831, 833, 837, 843, 867, 873, 875, 885, 887, 893, 911, 915, 923, 935, 945, 951, 963, 977, 981, 983, 987, 1001, 1005, 1007, 1013, 1023, 1025, 1035, 1053, 1055, 1061, 1065, 1071, 1085, 1091, 1113, 1121, 1127, 1131, 1137, 1151, 1161, 1163, 1173, 1181, 1197, 1233, 1247, 1251, 1253, 1263, 1265, 1275, 1277, 1287, 1293, 1295, 1307, 1331, 1337, 1347, 1355, 1361, 1371, 1377, 1383, 1391, 1413, 1415, 1421, 1445, 1457, 1461, 1463, 1467, 1481, 1485, 1497, 1503, 1511, 1517, 1523, 1533, 1541, 1551, 1553, 1575, 1581, 1595, 1607, 1613, 1623, 1637, 1641, 1643, 1655, 1665, 1673, 1691, 1695, 1701, 1707, 1715, 1721, 1725, 1727, 1737, 1743, 1755, 1757, 1761, 1797, 1803, 1805, 1811, 1821, 1827, 1833, 1853, 1863, 1865, 1875, 1877, 1881, 1887, 1901, 1911, 1923, 1925, 1931, 1967, 1971, 1977, 1991, 2003, 2015, 2031, 2037, 2043, 2045, 2055, 2061, 2073, 2091, 2097, 2105, 2111, 2117, 2135, 2141, 2157, 2175, 2177, 2183, 2187, 2195, 2201, 2205, 2217, 2231, 2241, 2243, 2253, 2261, 2265, 2267, 2273, 2297, 2301, 2307, 2321, 2331, 2343, 2351, 2355, 2367, 2373, 2385, 2387, 2397, 2405, 2411, 2435, 2447, 2451, 2453, 2463, 2465, 2471, 2477, 2481, 2483	316

Note: The four blocks of each member of the XXSW series can be plugged into the Goethals-Seidel array to obtain a skew Hadamard matrix. However the orders of these matrices are the same as those in the Paley-3 series.

Szekeres series (see [33]): 171 matrices, $v = (1+q)/2$, $q \equiv 5 \pmod 8$ a prime power. 461

Series	Cumulative Total
19, 31, 51, 55, 75, 79, 91, 99, 115, 135, 139, 159, 175, 187, 195, 199, 211, 231, 271, 279, 307, 331, 339, 351, 355, 367, 379, 387, 399, 411, 415, 427, 439, 471, 499, 511, 531, 535, 555, 559, 591, 607, 619, 639, 651, 687, 691, 715, 727, 747, 775, 799, 807, 811, 819, 835, 847, 855, 871, 895, 931, 939, 967, 975, 999, 1015, 1027, 1099, 1107, 1111, 1119, 1135, 1147, 1155, 1167, 1171, 1179, 1191, 1195, 1219, 1239, 1279, 1311, 1339, 1375, 1395, 1399, 1419, 1431, 1455, 1459, 1479, 1519, 1531, 1555, 1563, 1591, 1611, 1615, 1627, 1651, 1687, 1731, 1735, 1759, 1767, 1771, 1779, 1791, 1807, 1819, 1839, 1851, 1855, 1867, 1899, 1927, 1939, 1959, 1995, 2007, 2011, 2047, 2067, 2079, 2115, 2127, 2131, 2179, 2199, 2211, 2247, 2259, 2275, 2299, 2311, 2319, 2395, 2407, 2431, 2439, 2455, 2467, 2479, 2487	461

Szekeres-Whiteman series (see [34, 38]): 13 matrices, $v = (1+q)/2$, $q = p^k$, p mod 8=5 a prime and k even.

Series	Cumulative Total
13, 85, 313, 421, 685, 1405, 1861	468

Spence series (see [32]): 13 matrices, $v = 1 + q + q^2$, q is a prime not congruent to 1 mod 8 or $3 + 2q + 2q^2$ is a prime power.

Series	Cumulative Total
183, 757, 993, 1723	472

There are 41 sporadic values of v: 513

Series	Cumulative Total
9, 23, 25, 29, 37, 39, 43, 47, 49, 59, 65, 67, 69, 73, 81, 93, 97, 103, 109, 113, 121, 127, 129, 133, 145, 151, 157, 163, 169, 181, 189, 213, 217, 219, 239, 241, 247, 267, 397, 463, 631.	513

For $v = 9$ see [20], for $v = 23$ see [39], for $v = 25$ see [21], for $v = 29$ see [35], for $v = 37, 43$ see [7], for $v = 39, 49, 65, 93, 121, 129, 133, 217, 219, 267$ see [9], for $v = 47, 97$ see [11], for $v = 59$ see [18], for $v = 67, 113, 127, 157, 163, 189, 241$ see [8], for $v = 69, 73$ see [15],
for $v = 81, 103, 151, 169, 463$ see [10], for $v = 109, 145, 247$ see [12], for $v = 127, 191$ see [14], for $v = 189$ see the next section, for $v = 213, 631$ see [13], for $v = 239$ see [17], for $v = 397$ see [16].

Finally we list the 737 undecided cases of odd $v < 2500$:

89, 101, 107, 119, 149, 153, 167, 177, 179, 191, 193, 201, 205, 209, 223,
225, 229, 233, 235, 245, 249, 251, 253, 257, 259, 261, 265, 269, 275, 277,
283, 285, 287, 289, 295, 299, 301, 303, 305, 309, 311, 317, 319, 325, 329,
335, 337, 341, 343, 345, 347, 349, 353, 359, 361, 369, 373, 377, 385, 389,
391, 401, 403, 409, 413, 419, 423, 429, 433, 435, 441, 443, 445, 449, 451,
455, 457, 459, 461, 465, 469, 473, 475, 479, 481, 485, 487, 489, 491, 493,
505, 509, 513, 515, 517, 519, 523, 527, 529, 537, 539, 541, 543, 549, 553,
557, 565, 569, 571, 573, 575, 577, 579, 581, 583, 589, 595, 597, 599, 601,
605, 609, 611, 613, 621, 623, 625, 627, 629, 637, 641, 643, 647, 649, 653,
655, 657, 659, 661, 663, 667, 669, 673, 679, 681, 689, 693, 695, 697, 699,
703, 707, 709, 717, 719, 721, 723, 725, 729, 731, 733, 737, 739, 745, 749,
751, 759, 761, 763, 765, 769, 773, 777, 779, 781, 783, 785, 787, 789, 793,
795, 803, 805, 809, 817, 821, 823, 829, 839, 841, 845, 849, 851, 853, 857,
859, 861, 863, 865, 869, 877, 879, 881, 883, 889, 891, 897, 899, 901, 903,
905, 907, 909, 913, 917, 919, 921, 925, 927, 929, 933, 937, 941, 943, 947,
949, 953, 955, 957, 959, 961, 965, 969, 971, 973, 979, 985, 989, 991, 995,
997, 1003, 1009, 1011, 1017, 1019, 1021, 1029, 1031, 1033, 1037, 1039,
1041, 1043, 1045, 1047, 1049, 1051, 1057, 1059, 1063, 1067, 1069, 1073,
1075, 1077, 1079, 1081, 1083, 1087, 1089, 1093, 1095, 1097, 1101, 1103,
1105, 1109, 1115, 1117, 1123, 1125, 1129, 1133, 1139, 1141, 1143, 1145,
1149, 1153, 1157, 1159, 1165, 1169, 1175, 1177, 1183, 1185, 1187, 1189,
1193, 1199, 1201, 1203, 1205, 1207, 1209, 1211, 1213, 1215, 1217, 1221,
1223, 1225, 1227, 1229, 1231, 1235, 1237, 1241, 1243, 1245, 1249, 1255,
1257, 1259, 1261, 1267, 1269, 1271, 1273, 1281, 1283, 1285, 1289, 1291,
1297, 1299, 1301, 1303, 1305, 1309, 1313, 1315, 1317, 1319, 1321, 1323,
1325, 1327, 1329, 1333, 1335, 1341, 1343, 1345, 1349, 1351, 1353, 1357,
1359, 1363, 1365, 1367, 1369, 1373, 1379, 1381, 1385, 1387, 1389, 1393,
1397, 1401, 1403, 1407, 1409, 1411, 1417, 1423, 1425, 1427, 1429, 1433,
1435, 1437, 1439, 1441, 1443, 1447, 1449, 1451, 1453, 1465, 1469, 1471,
1473, 1475, 1477, 1483, 1487, 1489, 1491, 1493, 1495, 1499, 1501, 1505,
1507, 1509, 1513, 1515, 1521, 1525, 1527, 1529, 1535, 1537, 1539, 1543,
1545, 1547, 1549, 1557, 1559, 1561, 1565, 1567, 1569, 1571, 1573, 1577,
1579, 1583, 1585, 1587, 1589, 1593, 1597, 1599, 1601, 1603, 1605, 1609,

1617, 1619, 1621, 1625, 1629, 1631, 1633, 1635, 1639, 1645, 1647, 1649, 1653, 1657, 1659, 1661, 1663, 1667, 1669, 1671, 1675, 1677, 1679, 1681, 1683, 1685, 1689, 1693, 1697, 1699, 1703, 1705, 1709, 1711, 1713, 1717, 1719, 1729, 1733, 1739, 1741, 1745, 1747, 1749, 1751, 1753, 1763, 1765, 1769, 1773, 1775, 1777, 1781, 1783, 1785, 1787, 1789, 1793, 1795, 1799, 1801, 1809, 1813, 1815, 1817, 1823, 1825, 1829, 1831, 1835, 1837, 1841, 1843, 1845, 1847, 1849, 1857, 1859, 1869, 1871, 1873, 1879, 1883, 1885, 1889, 1891, 1893, 1895, 1897, 1903, 1905, 1907, 1909, 1913, 1915, 1917, 1919, 1921, 1929, 1933, 1935, 1937, 1941, 1943, 1945, 1947, 1949, 1951, 1953, 1955, 1957, 1961, 1963, 1965, 1969, 1973, 1975, 1979, 1981, 1983, 1985, 1987, 1989, 1993, 1997, 1999, 2001, 2005, 2009, 2013, 2017, 2019, 2021, 2023, 2025, 2027, 2029, 2033, 2035, 2039, 2041, 2049, 2051, 2053, 2057, 2059, 2063, 2065, 2069, 2071, 2075, 2077, 2081, 2083, 2085, 2087, 2089, 2093, 2095, 2099, 2101, 2103, 2107, 2109, 2113, 2119, 2121, 2123, 2125, 2129, 2133, 2137, 2139, 2143, 2145, 2147, 2149, 2151, 2153, 2155, 2159, 2161, 2163, 2165, 2167, 2169, 2171, 2173, 2181, 2185, 2189, 2191, 2193, 2197, 2203, 2207, 2209, 2213, 2215, 2219, 2221, 2223, 2225, 2227, 2229, 2233, 2235, 2237, 2239, 2245, 2249, 2251, 2255, 2257, 2263, 2269, 2271, 2277, 2279, 2281, 2283, 2285, 2287, 2289, 2291, 2293, 2295, 2303, 2305, 2309, 2313, 2315, 2317, 2323, 2325, 2327, 2329, 2333, 2335, 2337, 2339, 2341, 2345, 2347, 2349, 2353, 2357, 2359, 2361, 2363, 2365, 2369, 2371, 2375, 2377, 2379, 2381, 2383, 2389, 2391, 2393, 2399, 2401, 2403, 2409, 2413, 2415, 2417, 2419, 2421, 2423, 2425, 2427, 2429, 2433, 2437, 2441, 2443, 2445, 2449, 2457, 2459, 2461, 2469, 2473, 2475, 2485, 2489, 2491, 2493, 2495, 2497, 2499

4. The case $v = 189$

Our objective in this section is to outline the construction of a skew Hadamard matrix H of order $4 \cdot 189$. The construction is based on the theorem of J. Seberry [26, Theorem 7]. Although such a matrix has been constructed recently by Cati and Pasechnik [5], they did not provide in their paper any details of the construction.

Two main ingredients that we need are amicable orthogonal designs (AOD) X and Y with parameters $(28 : (1, 27), (28))$ and an orthogonal design (OD) Z with parameters $(28 : 1, 1, 26)$.

The required AOD, X and Y, exist and can be constructed by using [27, Theorem 5.12]. Let $q = 27$ and let a, b, c be commuting variables. We have

$$X = \begin{bmatrix} a & b & \cdots & b \\ -b & & & \\ \vdots & & aI_q + bP & \\ -b & & & \end{bmatrix}, \quad Y = c \cdot \begin{bmatrix} -1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & (I_q + P)R_1 & \\ 1 & & & \end{bmatrix},$$

where I_q, P, R_1 are matrices of order q, I_q the identity, and R_1 the direct sum of [1] and the back diagonal permutation matrix of order $q - 1$. The matrix P is the so called Paley core. To be specific, let F_q be the finite field of order q constructed by using the irreducible polynomial $x^3 - x - 1 \pmod 3$. The element $g = x^2 - x - 1$ of F_q generates the multiplicative group of F_q. We arrange the elements of F_q as follows:

$$0, g^0, g^1, \ldots, g^{11}, g^{12}, -g^{12}, -g^{11}, \ldots, -g^1, -g^0$$

and use them as labels for rows and columns of P. Then we have $P[s,t] = \chi(s-t)$ for all $s,t \in F_q$, where χ is the quadratic character of F_q. For convenience we set $D = (I_q + P)R_1$. One can easily verify that D is symmetric, i.e. $D = D^T$ where T denotes transposition. Further we have

$$XX^T = (a^2 + 27b^2)I_{q+1}, \; YY^T = 28c^2 I_{q+1}.$$

The required OD, Z, also exists. Indeed from Table C.1, on p. 380 of [27] an OD(28: 1,1,1,25) is specified by the quadruple
addd̄dd̄d̄; bddd̄dd̄d̄; cddd̄dd̄d̄; d̄dddddd.
To get an OD(28: 1,1,26) we just set $d = c$ and obtain the quadruple
accc̄cc̄c̄; bccc̄cc̄c̄; cccc̄cc̄c̄; c̄cccccc.
The letter c̄ stands for $-c$. Hence we obtain the four sequences:
$a, c, c, -c, c, -c, -c$; $b, c, c, -c, c, -c, -c$; $c, c, c, -c, c, -c, -c$; $-c, c, c, c, c, c, c$.
Next form the four circulant matrices A_0, A_1, A_2, A_3 of order 7 whose first rows are these four sequences respectively. Finally insert the circulants A_0, A_1, A_2, A_3 into the Goethals-Seidel array

$$\begin{bmatrix} A_0 & A_1R & A_2R & A_3R \\ -A_1R & A_0 & -RA_3 & RA_2 \\ -A_2R & RA_3 & A_0 & -RA_1 \\ -A_3R & -RA_2 & RA_1 & A_0 \end{bmatrix}$$

where R denotes the back diagonal permutation matrix of order 7. The resulting matrix Z of order 28 is the required OD(28: 1,1,26). Indeed, one can verify that $ZZ^T = (a^2 + b^2 + 26c^2)I_{28}$.

The final step is to replace the variables a, b, c in Z by the matrices $I_q + P, J, D$ respectively, where J is the matrix of order 27 with all entries equal 1. Then one can verify that the resulting matrix H of order $27 \cdot 28 = 756 = 4 \cdot 189$ is indeed a skew Hadamard matrix.

References

[1] L.V. Abuzin, N.A. Balonin, D.Ž. Đoković and I.S. Kotsireas, Hadamard matrices from Goethals-Seidel difference families with a repeated block, Information and Control Systems 2019, no. 5, pp. 2-9.

[2] N.A. Balonin, D.Z. Djokovic, Symmetric Hadamard matrices of orders 268, 412, 436 and 604, Information and Control Systems 4(95)/2018, 2-7.

[3] N.A. Balonin, D.Ž. Đoković, and D.A. Karbovskiy, Construction of symmetric Hadamard matrices of order $4v$ for $v = 47, 73, 113$, Spec. Matrices 6 (2018), 11-22.

[4] V. Belevitch, Conference networks and Hadamard matrices, Ann. Soc. Sci. Brux. 82 (1968), 13-32.

[5] Matteo Cati and Dmitrii V. Pasechnik, A database of constructions of Hadamard matrices, arXiv 2411.18897v1.

[6] O. Di Mateo, D.Z. Djokovic, I.S. Kotsireas, Symmetric Hadamard matrices of order 116 and 172 exist, Special Matrices, 2015, no. 3, pp. 227-234.

[7] Dragomir Ž. Đoković, Skew Hadamard matrices of order 4 x 37 and 4 x 43, J. Combin. Theory, Ser. A 61 (1992), 319-321.

[8] Dragomir Ž. Đoković, Construction of some new Hadamard matrices. Bull. Austral. Math. Soc. 45 (1992), 327-332.

[9] Dragomir Ž. Đoković, Ten new orders for Hadamard matrices of skew type. Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. 3 (1992), 47-59.

[10] Dragomir Ž. Đoković, Five new orders for Hadamard matrices of skew type. Australas. J Combin. 10 (1994), 259—264.

[11] Dragomir Ž. Đoković, Skew-Hadamard matrices of orders 188 and 388 exist. International Mathematical Forum, 3, 22 (2008), 1063-1068.

[12] Dragomir Ž. Đoković, Skew-Hadamard matrices of orders 436, 580, and 988 exist, J. Combin. Designs, 16 (2008), 493-498.
[13] D.Ž. Đoković, O. Golubitsky and I. S. Kotsireas, Some new orders of Hadamard and skew-Hadamard matrices. J. Combin. Designs, 22 (2014), 270-277.
[14] Dragomir Ž. Đoković, Some new symmetric Hadamard matrices, Information and Control Systems 2022, no. 2, pp. 2-9.
[15] Dragomir Ž. Đoković, Hadamard matrices: skew of orders 276, 292 and symmetric 372, arXiv 2301.02751v3.
[16] Dragomir Ž. Đoković, Two classes of Hadamard matrices of Goethals-Seidel type, arXiv 2404.14375v3.
[17] D.Ž. Đoković, I.S. Kotsireas, A class of cyclic $(v; k_1, k_2, k_3; \lambda)$ difference families with v = 3 (mod 4) a prime. Special Matrices 2016: 4:317-325.
[18] R.J. Fletcher, C. Koukouvinos, J. Seberry, New skew-Hadamard matrices of order $4 \cdot 59$ and new D-optimal designs of order $2 \cdot 59$, Discrete Mathematics 286 (2004), 251-253.
[19] A.V. Geramita and J.H. Verner, Orthogonal designs with zero diagonal, Can. J. Math. XXVIII, No. 1, (1976), 215-224.
[20] J. M. Goethals and J. J. Seidel, A skew-Hadamard matrix of order 36, J. Austral. Math. Soc. 11 (1970), 343-344.
[21] D.C. Hunt, Skew-Hadamard matrices of order less than 100, In Proceedings of the First Australian Conference on Combinatorial Mathematics, 23–27.
[22] R. Mathon, Symmetric conference matrices of order $pq^2 + 1$, Canad. J. Math. 30 (1978), 321-331.
[23] M. Miyamoto, A construction of Hadamard matrices, J. Combin. Theory, Ser. A 57, (1991), 86-108.
[24] M. Muzichuk, Q. Xiang, Symmetric Bush-type Hadamard matrices of order $4m^4$ exist for all odd m, Proc. Amer. Math. Soc. 134 (2006) no. 8, 2197-2204.
[25] R.E.A.C. Paley, On orthogonal matrices, J. Math. Phys. 12 (1933), 311-320.
[26] J. Seberry, On skew-Hadamard matrices, Ars Combinatoria, vol. 6 (1978), 255-276.
[27] J. Seberry, Orthogonal Designs: Hadamard Matrices, Quadratic Forms and Algebras, Springer Nature, Chaum, 2017.
[28] J. Seberry, N.A. Balonin, Two infinite families of symmetric Hadamard matrices, Australasian J. Combinatorics, vol. 69(3) (2017), 349-357.
[29] J. Seberry, A.L. Whiteman, New Hadamard matrices and conference matrices obtained via Mathon's construction, Graphs Combin. 4 (1988), 355-377.
[30] J. Seberry, M. Yamada, Hadamard matrices, sequences, and block designs, in Contemporary Design Theory, A Collection of Surveys (J. H. Dinitz and D. R. Stinson, Eds.) pp. 431-560. John Wiley and Sons, New York, 1992.
[31] J. Seberry, M. Yamada, Hadamard Matrices, Constructions using Number Theory and Algebra, John Wiley & Sons, 2020.
[32] E. Spence, Skew-Hadamard matrices of Goethals-Seidel type, Canad. J. Math. 27 (1975), 555-560.
[33] G. Szekeres, Tournaments and Hadamard matrices, Enseignement Math. 15 (1969), 269-278.
[34] G. Szekeres, Cyclotomy and complementary difference sets, Acta Arithmetica 18 (1971), 349-353.
[35] G. Szekeres, A note on skew type orthogonal ±1 matrices, In Combinatorics, Colloquia Mathematica Societatis Janos Bolyai, 52 (1988), 489-498.
[36] R.J. Turyn. On C-matrices of arbitrary powers. Can. J. Math. 23 (1971), 531-535.
[37] R.J. Turyn, A special class of Williamson matrices and difference sets, J. Combin. Theory, Ser. A, 36 (1984), 111-115.
[38] A.L. Whiteman, An infinite family of skew-Hadamard matrices, Pacific J. Math. 38 (1971), 817-822.
[39] J.S. Wallis, A skew-Hadamard matrix of order 92, Bull. Austral. Math. Soc. 5 (1971), 203-204.
[40] M. Xia, T. Xia, J. Seberry, J. Wu, An infinite family of Goethals-Seidel arrays, Discrete Applied Mathematics 145 (2005), 498-504.

Email address: DRAGOMIR@ROGERS.COM
Email address: J.SEBERRY@UOW.EDU.AU